Default Risk Model in a Fuzzy Framework

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Abstract. Default risk model is provided by using option pricing theory in a fuzzy framework in consideration of a simple company comprised of a single type of the debt that is free from profit payment and a single type of capital that is liberated from dividend. The model is based on the assumption that asset value of a company is the sum of total market value of stock and debt value, considering a situation where the asset value becomes below the debt value is default. For constructing the default risk model, a new variable is defined to derive a formula to evaluate the probability of default. Thus, a EDP(estimated default probability) model with the first and second moment is proposed and since debt value is fluctuated as asset price some bounds are established to somehow admit the fluctuation of the debt values, employing fuzzy number in the total market value of asset value.

Keywords: Default risk model, Total market value, Stock price, Debt value, Moments, Fuzzy numbers, Black-Scholes-Merton model

1 Introduction

On September 15 of 2008, Lehman Brothers, one of the largest investment banks, declared bankruptcy, having failed to find an investor, a buyer, or government guarantees. The failure of Lehman and the government's determination not to rescue the firm sharply raised investor fears. Since then this world has suffered from a financial crisis. Severity of cumulative debts has been deepened and risk control of increasing derivative transactions has become a problematic point. Thus, it becomes very important not only for financial institutions but also for general companies in possession of stocks or bonds to evaluate management abilities, credit power of companies in an fair and simple manner. Especially with the financial institutions including banks, soundness in management is required in performing international business endeavors and in making transactions competitive enough to cope with foreign financial institutions in a field of international competitions. In controlling such a kind of a credit risks, an estimation method of the default probability for the clients with whom credit should be provided becomes important. Proper estimation of the default probability becomes very important information when observation is made with proper pricing of the bond values to which credit risks are applied or with adequate profit brought about by market transactions, resulting in foundation to evaluate these factors.

As a representative method of structural approach taken up in this study, Merton[2], who stipulates default as a situation where asset values become below debt values as default, modeled default risks of the bonds by an option pricing theory. Black and Cox[3] developed a model that default is caused at the time when the value of the profit of the company reaches a certain low threshold value by relaxing the assumption based on a framework of the Merton model [2] as a foundation. However, Merton or Black-Scholes model [1,2](B-S-M model) may not be appropriate to evaluate default probability since the EDP value is occasionally underestimated for heightening of volatility usually caused by critical affairs of the company and increase of a ratio of the leverage. The model assumes the feature that default occurs only when a company runs through its all assets. On the other hand, it is not straightforward to estimate asset value in obtaining EDP since the asset value itself cannot be directly observed in the market and the conventional methods allow to using nonlinear equation with stock price data, giving complicated calculations.

Miyake and Inoue [9] propose a new methodology in place of these to estimate asset value by taking advantage of moments. Levy[6] uses moments for Asian option pricing and Inoue-Miyake-Takahashi-Yu [7] derive weighted Asian rate option, then it is applied to strike type by Miyake and Inoue [8]. Miyake-Inoue assume that asset value as the sum of the total market value of current stock and debt value , and propose a default model based on moments to estimate its probability, showing its adequacy with the application of Japanese companies.

While the total market value is easily found with multiplying stock value daily observed in the market by the number of issued total stocks, for total market value of debt is difficult to be observed in the market. In evaluation EDP of companies, Ronn and Verma [10] estimates debt value from the data of stock value, and Ando and Marushige [12] use book value of the debt. However, the debt value as well as stock price actually shows fluctuation whose trend is not regularly or systematically.

In this study, we propose a methodology to estimate default probability with the first and second moments in fuzzy approach in which debt value in repayment time may be described with fuzzy numbers.

2 Default Risk Model of B-S-M

Default risk model is provided by using an option pricing theory by the B-S-M model in consideration of a simple company comprised of a single type of the debt that is free from profit payment and a single type of capital that is liberated from dividend. This model is based on the assumption that a situation where asset value becomes below the debt is default. Then the assumption below is made with the EDP estimation by means of the B-S-M model. The asset value of a company follows a stochastic process below,

$$dA_\tau = \mu_A A_\tau d\tau + \sigma_A A_\tau dW_\tau \tag{1}$$

where μ_A is the expected profit ratio of the asset value, σ_A is the volatility of the asset value and dW_τ is standard Brownian motion. From (1) the asset value A_τ of the $\tau(0 \le \tau \le T)$ time point can be expressed for A_t .

$$A_\tau = A_0 \exp((\mu_A - \sigma_A^2 / 2)\tau + \sigma_A W_\tau), \quad (2)$$

where $W_\tau \sim N(0,\tau)$. Denoting the debt value at the time point τ by D_τ, the default event can be expressed as $\{A_\tau \le D_\tau\}$. The default probability EDP on the Black-Scholes-Merton model is given below.

$$\begin{aligned} EDP &= P(\ln A_\tau \le \ln D_\tau) \\ &= 1 - N\left(\frac{\ln(A_0 / D_\tau) + (\mu_A - \sigma_A^2 / 2)\tau}{\sigma_A \sqrt{\tau}}\right) \end{aligned} \quad (3)$$

where $N(\cdot)$ is an accumulated probability density function of the standard normal distribution.

In this study, we make the following assumptions for the four parameters as shown below. 1) Let the asset value A_t be the sum of the total market value of current stock E_0 and debt value D_0 2) The debt value cannot be observed in the market, and let it be understood that D_τ is book value of the debt 3)Let the expected profit ratio of the asset value be the risk-free interest rate.

Denote volatility of the total market value of current stock by σ_E . On the assumption of 1), the expression below is obtained from Itô 's lemma because the volatility σ_E is observable on the market. That is,

$$\sigma_E E_0 = N(x)\sigma_A A_0 \quad (4)$$

where

$$x = \frac{\ln(A_0 / D_\tau) + (r + \sigma_A^2 / 2)\tau}{\sigma_A \sqrt{\tau}}$$

3 Default Risk Model with the First and Second Moments

We consider the first and second moments of the asset value and by defining a new variable we may take the following procedure.

a) The first and second moments such as a mean value and variance of the sum of the total value of stock price and debt value together are derived.
b) Then, a new variable X following geometric Brownian motion, where fluctuation in the model evaluation period coincides with the moments of the sum of the total value of stock price and debt value, is assumed.

c) After obtaining the first and second moments of the new variable and by letting these moments be equal of those obtained in a) we can find the expected profit ratio and volatility with respect to the new variable X .

According to Merton[2], the asset value is expressed as the sum of the total market value of stock price and debt value. While the total market value is easily found by multiplying stock value daily observed in the market by the number of issued total stocks, for total market value of debt is difficult to be observed in the market. On the assumption that total market value of current stock price follows geometric Brownian motion, the total value of current price of stock E_τ can be expressed.

$$E_\tau = E_0 \exp((\mu_E - \sigma_E^2 / 2)\tau + \sigma_E W_\tau) \tag{5}$$

where μ_E, σ_E are the expected profit ratio and volatility of total market values. Miyake-Inoue [8] estimate EDP by using book value of the debt as substitute for debt value for valuing debt. As mentioned above the total market value of stock price is expressed as

$$A_\tau = E_\tau + D_\tau \tag{6}$$

The first and second moments of A_τ are obtained below

$$\begin{aligned} E[A_\tau] &= E_0\, e^{\mu_E \tau} + D_\tau \\ E[A_\tau^2] &= E_0^2 e^{(2\mu_E + \sigma_E^2)\tau} + 2E_0 D_\tau\, e^{\mu_E \tau} + D_\tau^2 \end{aligned} \tag{7}$$

Next, assume a new variable X follows stochastic process with Brownian motion whose expected profit ratio and volatility may be obtained in (9) and (10).

$$dX_\tau = \mu_x X_\tau dt + \sigma_x X_\tau dW_\tau \tag{8}$$

Since we assume that the fluctuation of the new variable during the evaluation period of the variable coincides with the first and second moments of the total value of stock price and debt value, letting the first and second moments be equivalent to those obtained above, the drift ratio and volatility can be obtained as below,

$$\mu_x = \frac{1}{\tau} \ln\left(\frac{E_0\, e^{\mu_E \tau} + D_\tau}{X_0} \right) \tag{9}$$

and

$$\sigma_x = \sqrt{\frac{1}{\tau} \ln\left(\frac{E_0^2 e^{(2\mu_E + \sigma_E^2)\tau} + 2E_0\, D_\tau e^{\mu_E \tau} + D_\tau^2}{X_0^2} \right) - 2\mu_x} \tag{10}$$

By defining default as an event that asset value X_τ becomes below debt value D_τ, default probability is obtained below.

$$EDP = P(X_\tau < D_\tau) = 1 - N\left(\frac{\ln(X_0 / D_\tau) + (\mu_x - \sigma_x^2 / 2)\tau}{\sigma_x \sqrt{\tau}} \right) \quad \textbf{(11)}$$

Remark1. Comparison of B-S-M model and our model with moments

Look at EDP transition of the proposed model and conventional model for Mical which is a retail business company and became a default company on September 1st of 2001. Table 1 shows that a considerable amount of dissociation is seen with the EDP of both models in the vicinity where default was made.

Table 1

	2001/08/29	2001/09/06	2001/09/14	2001/09/17	2001/09/25
Proposed model	23.059%	25.163%	26.835%	45.632%	94.092%
B-S-M model	20.536%	22.121%	22.694%	34.997%	40.558%

The default point comes up to 26% with the proposed model, and this becomes 94.1% in October 2001 to be increased to 98.0% in December 2001. Contrary to the above, the EDP of the default point reaches 22.7% in case of the conventional model. On October 2001, the EDP becomes 39.2% to be decreased to 21.7% in December 2001. Thus contradictory behavior is noticed as seen for example in decrease of the EDP after the default. For more other examples see Miyake and Inoue [9]. Fig.1 shows that though both the models indicate similar movement with EDP the proposed model remarkably improves EDP in the vicinity of default point, representing the results fitted and accepted for more actual phenomenon.

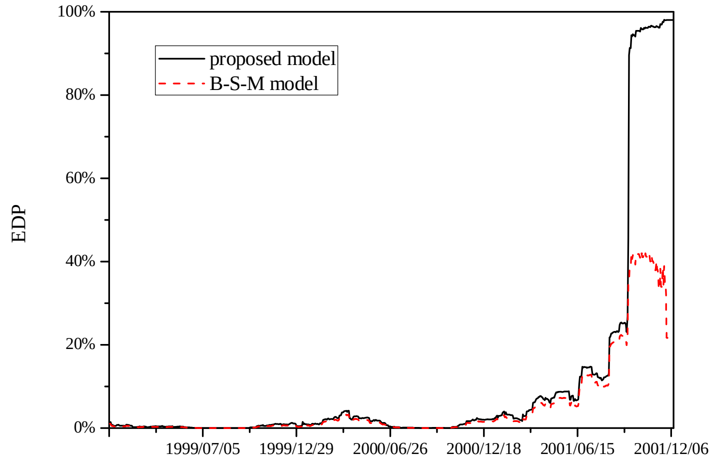

Fig. 1. EDP for B-S-M model and proposed model

4 Default Risk Model with Fuzzy Nature

So far, we have tried to estimate the default probability of a company with replacing debt value by book value of debt. However, note that the debt value as well as the stock price may be fluctuated during the evaluation period. Then, it is expected for some bounds to somehow admit the fluctuation of the debt values, incorporating fuzzy nature in the total market value of asset value.

For debt value D_τ at the period τ, we consider triangular fuzzy number $\tilde{D}_\tau = ((\tilde{D}_\tau)_L, (\tilde{D}_\tau)_C, (\tilde{D}_\tau)_R)$. The membership function of the triangular fuzzy number $\tilde{D}_\tau$ is defined below.

$$\varphi_{\tilde{D}_\tau}(x) = \begin{cases} 0, & x < (\tilde{D}_\tau)_L \\ (x - (\tilde{D}_\tau)_L)/((\tilde{D}_\tau)_C - (\tilde{D}_\tau)_L), & (\tilde{D}_\tau)_L < x < (\tilde{D}_\tau)_C \\ ((\tilde{D}_\tau)_U - x)/((\tilde{D}_\tau)_U - (\tilde{D}_\tau)_C), & (\tilde{D}_\tau)_C < x < (\tilde{D}_\tau)_U \\ 0, & x > (\tilde{D}_\tau)_U \end{cases} \quad \textbf{(12)}$$

The α -level set of $\tilde{D}_\tau$ is then, for any $\alpha \in [0,1]$

$$(\tilde{D}_\tau)_\alpha = [(\tilde{D}_\tau)^L_\alpha, (\tilde{D}_\tau)^U_\alpha] \quad \textbf{(13)}$$

That is,

$$(\tilde{D}_\tau)^L_\alpha = (1-\alpha)(\tilde{D}_\tau)_L + \alpha(\tilde{D}_\tau)_C,\ (\tilde{D}_\tau)^U_\alpha = (1-\alpha)(\tilde{D}_\tau)_R + \alpha(\tilde{D}_\tau)_C$$

Thus, the asset value of the company firm is described.

$$X^*_\tau = E_\tau + E[\tilde{D}_\tau] \quad \textbf{(14)}$$

where $E[(\tilde{D}_\tau)_\alpha]$ indicates the expectation of the triangular fuzzy number. According to Carlsson and Fuller[13] $E[(\tilde{D}_\tau)_\alpha]$ is expressed as below

$$E[\tilde{D}_\tau] = (\tilde{D}_\tau)_C + \frac{(\tilde{D}_\tau)_R + (\tilde{D}_\tau)_L - 2(\tilde{D}_\tau)_C}{6} \quad \textbf{(15)}$$

Thus, the expected profit ratio of the firm with fuzzy nature is obtained by (8)

$$\mu_{x^*} = \frac{1}{\tau}\ln\left(\frac{E_0 e^{\mu_E \tau} + E[\tilde{D}_\tau]}{X^*_0}\right) \quad \textbf{(16)}$$

,and the volatility becomes, by (9),

$$\sigma_{x^*} = \sqrt{\frac{1}{\tau}\ln\left(\frac{E_0^2 e^{(2\mu_E + \sigma_E^2)\tau} + 2E_0 E[\tilde{D}_\tau] e^{\mu_E \tau} + E[\tilde{D}_\tau]^2}{(X^*_0)^2}\right) - 2\mu_{x^*}} \quad \textbf{(17)}$$

Next, assume the following stochastic process which has equivalent expected profit ratio and volatility as (16),(17) and follows geometric Brownian motion

$$dX_{\tau}^{*} = \mu_{x^*} X_{\tau}^{*} d\tau + \sigma_{x^*} X_{\tau}^{*} dW_{\tau} \quad \textbf{(18)}$$

By considering (18) as asset value and defining default as event that asset value becomes below the fuzzy number of debt value
$\tilde{D}_{\tau}$ at time τ the default risk model with fuzzy nature is obtained below

$$\begin{aligned}(E\tilde{D}P)_{\alpha} &= P(X_{\tau}^{*} \in (\tilde{D}_{\tau})_{\alpha}) \\ &= [(E\tilde{D}P)_{\alpha}^{L}, (E\tilde{D}P)_{\alpha}^{U}]\end{aligned} \quad \textbf{(19)}$$

where $(\tilde{D}_{\tau})_{\alpha}$ is an interval with level of (13). The left end point $(EDP)_{\alpha}^{L}$ and right end point $(EDP)^{U}$ become

$$(E\tilde{D}P)_{\alpha}^{L} = 1 - N(\tilde{d}_{\alpha}^{U}), \quad (E\tilde{D}P)_{\alpha}^{U} = 1 - N(\tilde{d}_{\alpha}^{L}) \quad \textbf{(20)}$$

where $\tilde{d}_{\alpha}^{L}, \tilde{d}_{\alpha}^{U}$ are

$$\tilde{d}_{\alpha}^{L} = \frac{\ln(X_0^{*} / (\tilde{D}_{\tau})_{\alpha}^{U}) + (\mu_{x^*} - \sigma_{x^*}^{2}/2)\tau}{\sigma_{x^*}\sqrt{\tau}}, \quad \tilde{d}_{\alpha}^{U} = \frac{\ln(X_0^{*} / (\tilde{D}_{\tau})_{\alpha}^{L}) + (\mu_{x^*} - \sigma_{x^*}^{2}/2)\tau}{\sigma_{x^*}\sqrt{\tau}}$$

5 Simulation Results with Different α Values

Figure 2,3 and 4 show intervals with respect to EDP obtained in previous section for α=0.6, 0.8, 1.0. With different α values transition of EDP is observed for a default company Mical illustrated in Remark1 of section 3. For α =0.6 or 0.8 it does not seem to suggest any pattern which may lead to be associated with the figure for α =1.0.

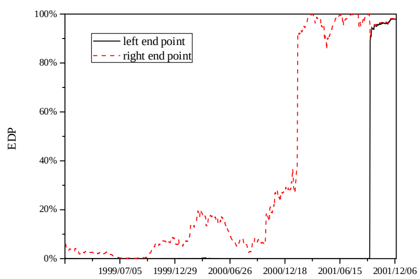

Fig. 2. EDP for α＝0.6

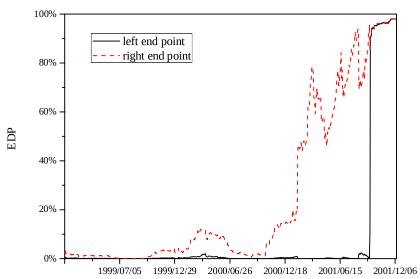

Fig. 3 EDP for α＝0.8

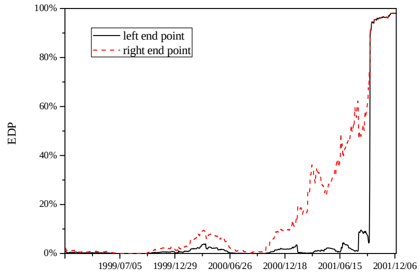

Fig. 4. EDP for α=0.9

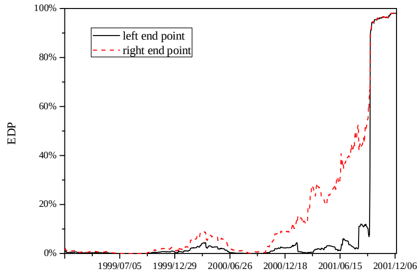

Fig. 5 EDP for α=0.92

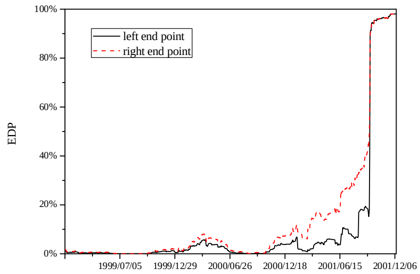

Fig. 6. EDP for α=0.96

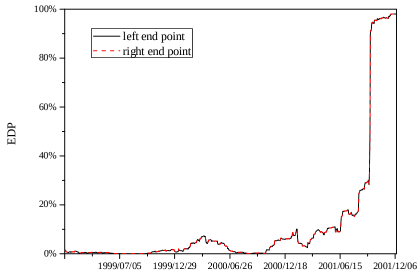

Fig.7. EDP for α=1.0

We note that there are two types of debt, one is current liabilities and the other fixed liabilities. In this example, for fuzzy number $\tilde{D}_\tau$ the left end point $(\tilde{D}_\tau)_L$ is calculated as current liabilities +0.5×fixed liabilities and the right end point $(\tilde{D}_\tau)_R$ as current liabilities×1.5 + fixed liabilities. The implication is interpreted as follows, for lower limit the liabilities don't increase and the fixed liabilities usually remains more than one year before its repayment time, which allows less fluctuation. On the contrary, the liabilities allow to increase for some reason during the evaluation period. For example, at α=0.6 the difference between lower limit and upper limit spreads to 58.33% on June 15 of 2001, but at α=0.8the difference makes decreased to 25.48%, and finally it becomes 0 at α=1.0.

6 Concluding Remarks

In this study, default risk model was presented in which the first and second moments are incorporated in the model to be sure of more preciseness so that the estimated probability for default gets higher than the conventional model by Merton [2]. The reason exists in that companies approaching to default situation show higher values of D/A, and the volatility of both the model plays a crucial role. Thus, the respective volatility σ_x and σ_A in EDP formulas dominates each of the whole expression of normal distribution and σ_x^2 is larger than σ_x^2 so that the above-mentioned may be understood.

However, the method is not sufficient in the light of the fluctuation of debt value and the debt value in the model remains unchanged, and hence is needed to admit the fluctuation in a sense of fuzzy approach. The additional work allows us to be able to cope with different circumstances we may face in the vicinity of default point. Thus, the debt value which may be expected at maturity time is treated with fuzzy feature, providing allowable intervals for the estimated probability of default with optimistic and pessimistic views.

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