OPTION PRICING FOR WEIGHTED AVERAGE OF ASSET PRICES

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Abstract

Average options are path-dependent and have payoffs which depend on the average price over a fixed period leading up to the maturity date. This option is of interest and important for thinly-traded assets since price manipulation is prohibited, and both the investor and issuer may enjoy a certain degree of protection from the caprice of the market. There are several nice results for the average options with different approaches. In this paper, we consider to propose a more general weight instead of usual simple average, for which it may be possible to control the weights in the light of unexpected situation incurred. An analytical form of option pricing along with Monte Carlo simulations is obtained, and a numerical example is presented. In particular, our methodologies are based on Kemna & Vorst, Vorst and Levy, so that some comparisons for their results are made.

Keywords. Weighted average option, Weighted sums, Conditional expectation, Geometric average, Arithmetic average

1 INTRODUCTION

Concerning the study of option pricing F.Black, M.Sholes and R.Merton made a major breakthrough in the past and the Black-Sholes model is still
used in various situations, in particular, in European option pricing problems.

Average options are fully path-dependent and have payoffs which depend on the average price of the underlying asset over a fixed period leading up to the maturity date. Since no general analytical solution for the price of the average option is known, a variety of techniques have been developed to analyze average options. There is historically enormous literature devoted to work of this option. Among them Bergman(1981) studies average rate options but only considers options with a zero strike price. Kemna and Vorst(1990,1992) propose a Monte Carlo methodology which employs the corresponding geometric option as a control variable. Carverhill and Clewlow (1990) use the Fourier Transformation to evaluate numerically the necessary convolutions of density functions. Ruttien (1990) and Vorst (1990) employing the solution to the corresponding geometric average problem improves the speed of calculation. On the other hand, Levy(1992) proposes a more accurate method which relies on the assumption that the distribution of sum of log normal variables is itself well approximated at least to a first order by the log normal. Also, the author assumes that the valuation of average option becomes possible for typical range of volatility experienced. The idea of such options are of particular interest and importance for thinly-traded assets since price manipulation is inhibited. This option is in general considered path dependent options for which the payoff at maturity date depends on the history of the prices the underlying asset takes.

In this paper, we are interested in pricing for which options payoff depends on the weighted sums of prices but not on simply the average price of underlying assets. As Kemna and Vorst mentioned in [1] that if a standard European call option is based on an asset which remains low in price during a large part of the final time period and rises significantly at maturity, the firm would not have been able to generate sufficient revenues to pay the high premium to the option. In that situation a pricing method based on standard asset values may not be sufficient so that more general weighted average option rather than the simple average option will be seriously considered. But, as Henderson(2004) pointed out, we don’t know to what stochastic process the average of exchange over a fixed period follows even though the path of exchange follows Brownian motion. Therefore, it will be needed to find probability distribution for the underlying asset and the prices of it. Thus, we extend the results for average options to pric-
ing problems based on the weighted sums of prices. Average option has
usually three different variables as explained later so that the problem is
how to solve three dimensionally stochastic differential equations. Wilmot
et al.(2002) reduce three to two dimensional stochastic differential equations
and use transformation of variables, applying Black-Sholes formula. Vecaf
et al.(2001) and Henderson(2004) assume that asset price follows submarting-
gele. Then, submartingale is considered as sum of martingale and increasing
process and hence it is possible to apply Black-Sholes formula even though
the stochastic process about average of exchange is not known.

This paper is organized as follows. Section 2 provides a brief description
of simple average option. In Section 3 a more general weighted average op-
tion is proposed and the analytical form is obtained. Some corrections for
arithmetic and geometric valuations are made with numerical examples in
Section 4. Section 5 provides another valuation formula based on moments
for arithmetic weighted option. In Section 6 these methodologies developed
in the previous sections are compared with numerical examples. Concluding
remarks are given in Section 7.

2 SIMPLE AVERAGE OPTION

We assume that a perfect security market which is open continuously,
offers a constant riskless interest rate $r$ to borrowers and lenders, in which
no transaction costs and / or taxes are incurred.

Let the underlying asset price $S(t)$ at time $t$ follow the geometric Brow-
nian motion process

$$dS(t) = \mu S(t) dt + \sigma S(t) dW(t),$$

where $dW(t)$ stands for a Wiener process which has a normal distribution
with the mean 0 the variance $dt$, $\mu$ is the drift parameter and $\sigma$ is the
volatility parameter.

The standard partial differential equation for the option price $C$ can be
derived by hedging arguments([10],[11]):

$$C_t + \frac{1}{2} \sigma^2 S^2 C_{ss} + r(SC_s - C) = 0$$

where $C_t$, $C_s$ are the first order partial derivatives with respect to $t$ and $S$
and $C_{ss}$ a second order partial derivative with respect to $S$. 

3
For \( t \in [T_0, T] \) consider the variable \( A(t) \) as

\[
A(t) = \frac{1}{t - T_0} \int_{T_0}^{t} S(\tau)d\tau
\]

where \( T \) is the maturity date. Then \( A(t) \) is defined as the part of the final average up to time \( t \) and the payoff on the option can be expressed as \( \text{Max}(A(t) - K, 0) \), where \( K \) is the exercise price of the average option.

Kemna and Vorst derive the expression for the value of the average option

\[
C(S(t), A(t), t) = e^{-r(T-t)}E^\# [\text{Max}(A(t) - K, 0)]
\]

with appropriate boundary conditions, where \( E^\# \) indicates the conditional expectation with respect to \( S(t), A(t) \) and \( t \).

Also, log value of \( S(\tau) \) after \( \tau = T - T_0 \) period passed is expressed as

\[
\log S(\tau) = \log S(T_0) + (r - 1/2\sigma^2)\tau + \sigma W(\tau)
\]

where \( \sigma W(\tau) \sim N(0, \sigma^2\tau) \).

There are two types of defining geometric average formulae to find the average prices of the asset, which are continuous and discrete cases;

\[
S_C = \exp \left( \frac{1}{T - T_0} \int_{T_0}^{T} \log S(\tau)d\tau \right)
\]

\[
S_D = \left( \prod_{i=1}^{N} S(T_i) \right)^{1/N}
\]

Under the assumption that the asset prices follow lognormal distribution the distribution of average price does not have desirable nature to be analyzed but geometric average possesses, and hence some adjustment will be needed to obtain more appropriate and reasonable option pricing. If asset prices are taken as just simple average its variation the asset possesses are almost lost.

We are interested in the situation for which a European call option based on an asset remains low in price during a large part of the final time period and rises significantly at maturity. In such a circumstance the firm would cope with this problem for some way.
3 A PRICING FORMULA FOR WEIGHTED AVERAGE OPTION

Thus, we want to propose more general way, in which weighted average rather than the simple average to control its variation is considered. In this study, we find a way to obtain the weighted sums of prices which depend on each asset price \( S(T_i) \) at time \( T_i \). Let the weight and the exercise price at \( T_i \) be \( W_i, K \), respectively. Denote by \( T_i = T_0 + ih \) for \( i = 1, 2, \cdots, N \) where \( h = (T - T_0)/N \). Define the weighted average

\[
A(T_m) = \sum_{i=1}^{m} W_i S(T_i) \quad \text{with} \quad \sum_{i=1}^{m} W_i = 1, \quad W_i > 0, i = 1, 2, \cdots, N
\]

for \( 1 < m < N \). Then, the weighted average option is characterized by the payoff function at time \( T_m \) given by \( \max(A(T_m) - K, 0) \) for a call and \( \max(K - A(T_m), 0) \) for a put option.

Over a fixed contract date to the maturity date \( T_m \) we consider the following weighted average defined below

\[
S_{AWA}(T_m) = \sum_{i=1}^{m} \frac{(1 + ah)^i}{\sum_{k=1}^{N} (1 + ah)^k} S(T_i), \quad -\infty < a < \infty
\]

In this expression a variable \( a \) to control the weights is incorporated and hence it may cope with more different situations incurred by underlying asset.

But the expression above is not suitable since each asset price \( S(T_i) \) is assumed to be log normally distributed and hence \( S_{AWA}(T_m) \) is a weighted sum of lognormally distributed. Then, \( S_{AWA}(T_m) \) is no longer lognormally distributed. Therefore, for a possible way to develop an approximation another type of weighted average option needs to be considered. The option based on geometric weighted average is

\[
S_{GWA}(T_m) = \prod_{i=1}^{m} S(T_i)^{w_i} \quad \text{with} \quad w_i = \frac{(1 + ah)^i}{\sum_{k=1}^{N} (1 + ah)^k} > 0, i = 1, 2, \cdots, N
\]

The final payoff at maturity of a call option on the geometric average is, denoting by \( C_{GWA} \) the values of the option,

\[
C_{GWA} = e^{-r(T - T_0)} E\# \left[ \max \left( S_{GWA}(T_m) - K, 0 \right) \right].
\]
Finally, the log value of geometric weighted average of (2) for continuous case is defined as

\[ \int_{T_0}^{T} e^{\alpha \tau} \log S(\tau) d\tau, \]

and hence \( \log S_{\text{GWA}}(T) \) is obtained

\[ \log S_{\text{GWA}}(T) = \frac{a}{e^{\alpha T} - e^{\alpha T_0}} \int_{T_0}^{T} e^{\alpha \tau} \log S(\tau) d\tau. \] (4)

**Remark 1** In the simple average, \( A(t) = \frac{1}{T - T_0} \int_{T_0}^{t} S(\tau) d\tau \) for \( T_0 \leq t \leq T \), the partial differential equation for the option price is derived by using arbitrage arguments, in which the factor relevant to \( A(t) \) has been incorporated \[1\], while the case of \( \log S_{\text{GWA}}(T) \) may not be possible. However, since the function of option price contains the variables, \( S(t), A(t) \) we are not able to find an explicit formula for the value of option.

**Remark 2** Kemna and Vorst show that when the price is based on the geometric average instead of the usual arithmetic average an analytic pricing formula can be derived, to gives a lower bound for the value of a call option based on the arithmetic average.

In finding the option prices for \( S_{\text{AWA}}(T) \) and \( S_{\text{GWA}}(T) \), since it is known that a geometric average is always lower than an arithmetic average \[14\], the prices obtained from the geometric average is usually underestimated so that some adjustments are needed to have suitable values. Since \( S_{\text{GWA}}(T) \) is a product of lognormally distributed variables it is also log normally distributed with the mean

\[ E[\log S_{\text{GWA}}(T)] = \log S(T_0) + \frac{(aT - 1) e^{\alpha T} - (aT_0 - 1) e^{\alpha T_0}}{a(e^{\alpha T} - e^{\alpha T_0})} \left( r - \frac{1}{2} \sigma^2 \right), \] (5)

and the variance is

\[ Var[\log S_{\text{GWA}}(T)] = \frac{(2aT - 3) e^{2\alpha T} + (2aT_0 - 1) e^{2\alpha T_0} - (4aT_0 - 4) e^{\alpha(T + T_0)}}{2a(e^{\alpha T} - e^{\alpha T_0})^2} \sigma^2. \] (6)
Thus, \( \log S_{\text{GWA}}(T) \) follows normal distribution

\[
\log S_{\text{GWA}}(T) \sim N \left( \log S(T_0) + \frac{(aT - 1)e^{aT} - (aT_0 - 1)e^{aT_0}}{a(e^{aT} - e^{aT_0})} \left( r - \frac{1}{2}\sigma^2 \right), \frac{(2aT - 3)e^{2aT} + (2aT_0 - 1)e^{2aT_0} - (4aT_0 - 4)e^{a(T + T_0)}}{2a(e^{aT} - e^{aT_0})^2} \right). 
\]

**Note:** When \( a \) is considered as a variable holding \( T - T_0 \) constant the expected values and variance above are obtained as \( a \to 0 \).

\[
E[\log S_{\text{GWA}}(T)] = \log S(T_0) + \frac{1}{2} \left( r - \frac{1}{2}\sigma^2 \right) (T - T_0)
\]

\[
\text{Var}[\log S_{\text{GWA}}(T)] = \frac{1}{3}\sigma^2(T - T_0).
\]

Then, following Jarrow and Rudd [9] and Vorst [2], we obtain the values for geometric weighted average for call and put options,

\[
C_{\text{GWA}} = e^{-r(T - T_0)} \left[ e^{d_{\text{C}} S(T_0)} N(d) - K N \left( d - \frac{\sigma}{e^{aT} - e^{aT_0}} \times \sqrt{\frac{(2aT - 3)e^{2aT} + (2aT_0 - 1)e^{2aT_0} - (4aT_0 - 4)e^{a(T + T_0)}}{2a(e^{aT} - e^{aT_0})^2}} \right) \right] 
\]

\[
P_{\text{GWA}} = e^{-r(T - T_0)} \left[ -e^{d_{\text{C}} S(T_0)} N(-d) + K N \left( -d + \frac{\sigma}{e^{aT} - e^{aT_0}} \times \sqrt{\frac{(2aT - 3)e^{2aT} + (2aT_0 - 1)e^{2aT_0} - (4aT_0 - 4)e^{a(T + T_0)}}{2a(e^{aT} - e^{aT_0})^2}} \right) \right] \]

where

\[
d^* = \frac{(aT - 1)e^{aT} - (aT_0 - 1)e^{aT_0}}{a(e^{aT} - e^{aT_0})} \left( r - \frac{1}{2}\sigma^2 \right) 
\]

\[
d^{**} = \frac{(2aT - 3)e^{2aT} + (2aT_0 - 1)e^{2aT_0} - (4aT_0 - 4)e^{a(T + T_0)}}{2a(e^{aT} - e^{aT_0})^2} \sigma^2 
\]

\[
d_C = d^* + \frac{1}{2}d^{**}, \quad d = \log \left( \frac{S(T_0)}{K} \right) + d^* + d^{**} \frac{1}{\sqrt{d^{**}}}
\]

Thus, we have obtained an analytic expression for \( C_{\text{GWA}} \) which satisfies our
requirements, and hence we can find a Monte Carlo estimate of the weighted average option to make some adjustments.

**Remark 3** On the other hand, the corresponding formulae for the standard average of geometric average option without control variable $a$ for a call and a put become

$$C_{GA} = e^{-r(T-T_0)} \left[ e^{dc} S(T_0) N(d) - KN \left( d - \sigma \sqrt{\frac{1}{3}(T-T_0)} \right) \right]$$

$$P_{GA} = e^{-r(T-T_0)} \left[ -e^{dc} S(T_0) N(-d) + KN \left( -d + \sigma \sqrt{\frac{1}{3}(T-T_0)} \right) \right]$$

where

$$d_C = \frac{1}{2} \left( r - \frac{1}{2} \sigma^2 \right) (T - T_0) + \frac{1}{6} \sigma^2 (T - T_0)$$

$$d = \frac{\log(S(T_0)/K) + \frac{1}{2} \left( r - \frac{1}{2} \sigma^2 \right) (T - T_0) + \frac{1}{2} \sigma^2 (T - T_0)}{\sigma \sqrt{\frac{1}{3}(T-T_0)}}$$

### 4 NUMERICAL COMPUTATIONS

#### 4.1 Adjustments For The Differences

There exist some differences in option premiums between arithmetic weighted average and geometric weighted average in Monte Carlo simulation so that we need to find the difference, adding it to the results $C_{GWA}$ which was based on (1) and (2). To proceed numerical computation the period $[0, T]$ is divided into $n$ subintervals. Then, it is distributed as a normal distribution with the mean $(r - \sigma^2/2)T/n$, variance $T/\sigma^2/n$. Therefore, the random sequence $S(T_1), ..., S(T_n)$ can be generated as follows

$$\log S(T_i) = \log S(T_{i-1}) + \left( r - \frac{1}{2} \sigma^2 \right) \frac{T}{n} + \sigma \sqrt{\frac{T}{n}} z_i,$$

where $z_i$ is assumed to be drawn from the standard normal distribution.

Then, the premium values for the arithmetic weighted sums and geometric weighted sums are given with (1) and (2)

$$\tilde{C}_{AWA} = e^{-r(T-T_0)} E^\# \{ \max(S_{AWA}(T_m) - K, 0) \}$$

$$\tilde{C}_{GWA} = e^{-r(T-T_0)} E^\# \{ \max(S_{GWA}(T_m) - K, 0) \}. \quad (8)$$
Thus, by Monte Carlo simulation the values of realizations of $\tilde{C}_{\text{AWA}}, \tilde{C}_{\text{GWA}}$ are calculated. Noting that a geometric average $S_{\text{GWA}}(T)$ is always less than an arithmetic average $S_{\text{AWA}}(T)$ [Beckenbach and Bellman, 1971] and hence $C_{\text{GWA}}$ is a lower bound for the call option value $C_{\text{AWA}}$. As a result, the following relation holds,

$$C_{\text{AWA}} \leq C_{\text{GWA}} + e^{-r(T-T_0)} \{ E^\# [S_{\text{AWA}}(T)] - E^\# [S_{\text{GWA}}(T)] \}$$

Therefore, the upper bound can be considered as a correction of the theoretical price for the difference between the expectations of the arithmetic average and the geometric average.

On the other hand, the proposed approximation implies to correct the exercise price $K$ for the difference these two expectations [Vorst, 1992]. The arithmetic average $S_{\text{AWA}}(T)$ for continuous case is expressed as

$$S_{\text{AWA}}(T) = \frac{a}{e^{aT} - e^{aT_0}} \int_{T_0}^{T} e^{a\tau} S(\tau) d\tau. \tag{9}$$

With some manipulation the expected value of $S_{\text{AWA}}(T)$ is derived

$$E^\# [S_{\text{AWA}}(T)] = \frac{aS(T_0)}{e^{aT} - e^{aT_0}} \left( \frac{e^{(r+a)T} - e^{(r+a)T_0}}{r + a} \right). \tag{10}$$

Note: As $a \to 0$ $E^\# [S_{\text{AWA}}(T)]$ becomes $S(T_0) \frac{e^{rT} - e^{rT_0}}{r(T-T_0)}$.

As a result, since the expected value of geometric average $E^\# [S_{\text{GWA}}(T)]$ is the form of (5), by replacing $K$ by $K - E^\# [S_{\text{AWA}}(T)] + E^\# [S_{\text{GWA}}(T)]$ we have

$$C_{\text{AWA}} = e^{-r(T-T_0)} E^\# \left[ \text{Max} \left( S_{\text{GWA}}(T) - (K - E^\# [S_{\text{AWA}}(T)] + E^\# [S_{\text{GWA}}(T)]) , 0 \right) \right].$$

In other words, the formulae for option price whose exercise price is adjusted are obtained by

$$C_{\text{AWA}} = e^{-r(T-T_0)} \left[ e^{\text{d}c} S(T_0) N(d) - K^* N \left( d - \frac{\sigma}{e^{aT} - e^{aT_0}} \times \sqrt{\frac{(2aT - 3) e^{2aT} + (2aT_0 - 1) e^{2aT_0} - (4aT_0 - 4) e^{a(T+T_0)}}{2a}} \right) \right],$$

9
\begin{align*}
P_{\text{AWA}} &= e^{-r(T-T_0)} \left[ -e^{dC} S(T_0) N(-d) + K^* N \left( -d + \frac{\sigma}{e^{aT} - e^{aT_0}} \right) \times \\
&\quad \sqrt{\frac{(2aT - 3)e^{2aT} + (2aT_0 - 1)e^{2aT_0} - (4aT_0 - 4)e^{a(T+T_0)}}{2a}} \right], \quad (11)
\end{align*}

where \( d_C \) and \( d \) are same as in (7) in the previous section, and

\begin{align*}
K^* &= K - \frac{aS(T_0)}{e^{aT} - e^{aT_0}} \left( \frac{e^{(r+a)T} - e^{(r+a)T_0}}{r + a} \right) \\
&+ S(T_0) \exp \left( \frac{(aT - 1)e^{aT} - (aT_0 - 1)e^{aT_0}}{a(e^{aT} - e^{aT_0})} \left( r - \frac{1}{2} \sigma^2 \right) \\
&+ \frac{(2aT - 3)e^{2aT} + (2aT_0 - 1)e^{2aT_0} - (4aT_0 - 4)e^{a(T+T_0)}}{4a(e^{aT} - e^{aT_0})^2} \right) \sigma^2. \quad (12)
\end{align*}

**Remark 4** In relation to **Remark 3**, the exercise price for standard average of geometric average option for a call and put is expressed as

\begin{align*}
K^* &= K - S(T_0) \frac{e^{rT} - e^{rT_0}}{r(T - T_0)} + S(T_0) \exp \left( \frac{1}{2} \left( r - \frac{1}{2} \sigma^2 \right) (T - T_0) + \frac{1}{6} \sigma^2 (T - T_0) \right).
\end{align*}

### 4.2 Numerical Example

First, option premiums for geometric weighted average based on section 3 and section 4 are compared with usual plain option when \( a \) is allowed to vary. Let \( S = 100 \), volatility = 20%, non-risk rate = 0.5%, \( T = 1 \) year. Simulation was performed 15000 times and observations 60 times.

It is observed from figure 1 through figure 3 that premium values derived by (7) show some changes, according as the variable \( a \) varies for different relationships of \( S \) and \( K \).

For \( S > K \) the premium values for geometric average for a call and a put are 7.3298, 2.4376, respectively, while those for plain options are 10.788 and 5.314, showing much difference for the both results. Also, note that the values for geometric weighted average, at \( a = 0 \), become equivalent to that for simple geometric average and become equivalent to that of plain option as \( a \to +\infty \). For \( S = K \) the similarity of the results are observed except that the premium value for call is slightly higher than that for put at any value of \( a \) as it is expected, and the opposite figure is also obtained for \( S < K \).
Figure 1: Premium values for $S > K (95)$

Figure 2: Premium values for $S = K (100)$

Figure 3: Premium values for $S < K (105)$
Next, the analytical expression obtained in previous section, (7) is used along with the Monte Carlo simulation to find appropriate premium values for call option. We obtain the option premiums for more weights placed on around maturity date and for more weights around contract date. Each value for geometric weighted average ($\tilde{C}_{GWA}$) and arithmetic average options ($\tilde{C}_{AWA}$) is computed for $T=6, 12$ and $\sigma=10\%, 20\%, 30\%$ and adding the difference to the value ($C_{GWA}$) calculated by analytical formula (7), obtaining the appropriate values of option ($C_A$). Table1 through Table3 show the results of this process.

It is observed that the results of $C_A$ for both cares are much smaller in standard deviations than those of Monte Carlo simulation.

Notice that the expression of exercise price $K^*$ used in (11) is different from that in (7), replacing by (12). The results are slightly higher than the analytical method.

5 A PRICING FORMULA BASED ON MOMENTS

As we worked above, options involving the arithmetic average may not have closed-form solutions if the conventional assumption of a geometric diffusion is specified for the underlying price process.

Levy[6] proposes a more accurate method which relies on the assumption that the distribution of sum of log normal variables is itself well approximated at least to a first order by the log normal. He mentions that the valuation of average option becomes possible for typical range of volatility experienced. Turnbull and Wakeman[15] also recognize the suitability of the log normal as a first-order approximation.

Thus, since first and second moments for arithmetic weighted average are possible to obtain we first define a new variable which follows lognormal distribution and corresponds up to first and second moments of the arithmetic weighted average. Then, option price obtained for such variable defined is nothing but that for plain option and hence Black-Scholes formula may be applicable.

The value of $S(\tau)$ after $\tau = T - T_0$ period passed is expressed as

$$S(\tau) = S(T_0) e^{\left(r-\frac{1}{2}\sigma^2\right)\tau + \sigma W(\tau)}$$

where $\sigma W(\tau) \sim N(0, \sigma^2 \tau)$. We also found the first moment of $S_{AWA}(T)$ as
(10),
\[ E [S_{AWA} (T)] = \frac{aS (T_0)}{e^{aT} - e^{aT_0}} \left( \frac{e^{(r+a)T} - e^{(r+a)T_0}}{r + a} \right). \]

Then, the second moment becomes
\[
E \left[ (S_{AWA} (T))^2 \right] = E \left[ \left( \frac{aS (T_0)}{e^{aT} - e^{aT_0}} \int_{T_0}^{T} e^{a\tau} S (\tau) d\tau \right)^2 \right] = 2 \left( \frac{aS (T_0)}{e^{aT} - e^{aT_0}} \right)^2 \left( \frac{e^{(2(r+a)+\sigma^2)T} - e^{(2(r+a)+\sigma^2)T_0}}{(2 (r + a) + \sigma^2) (r + a + \sigma^2)} - \frac{e^{(r+a)T_0} e^{(r+a+\sigma^2)T_0} - e^{(2(r+a)+\sigma^2)T_0}}{(r + a) (r + a + \sigma^2)} \right). \]

On the other hand, a new variable, \( X (T) \) we now define is considered to have the same price as that of underlying asset after \( T \) period. Assume \( \sigma W (\tau) \sim N (0, \sigma^2 \tau) \). Denoting \( r_x, \sigma_x \) by drift rate and volatility for the variable \( X (\tau) \), respectively, define
\[ X (\tau) = S (T_0) e^{(r_x - \frac{1}{2} \sigma_x^2) \tau + \sigma_x W (\tau)}. \]

Then, the first and second moments for \( X (T) \) can be expressed as
\[
E \left[ X (T) \right] = S (T_0) e^{r_x (T-T_0)}, \quad E \left[ (X (T))^2 \right] = (S (T_0))^2 e^{(2r_x+\sigma_x^2)(T-T_0)}. \]

These results along with (13), (14) give the expression
\[
S (T_0) e^{r_x (T-T_0)} = \frac{aS (T_0)}{e^{aT} - e^{aT_0}} \left( \frac{e^{(r+a)T} - e^{(r+a)T_0}}{r + a} \right).
\]

From this we obtain
\[
r_x = \frac{1}{T - T_0} \log \left( \frac{a}{e^{aT} - e^{aT_0}} \left( \frac{e^{(r+a)T} - e^{(r+a)T_0}}{r + a} \right) \right),
\]

and for \( \sigma_x \) we have
\[
(S (T_0))^2 e^{(2r_x+\sigma_x^2)(T-T_0)} = 2 \left( \frac{aS (T_0)}{e^{aT} - e^{aT_0}} \right)^2 \left( \frac{e^{(2(r+a)+\sigma^2)T} - e^{(2(r+a)+\sigma^2)T_0}}{(2 (r + a) + \sigma^2) (r + a + \sigma^2)} - \frac{e^{(r+a)T_0} e^{(r+a+\sigma^2)T_0} - e^{(2(r+a)+\sigma^2)T_0}}{(r + a) (r + a + \sigma^2)} \right). \]
Hence, we have
\[ \sigma_x = \sqrt{\frac{1}{T - T_0} \log A - 2r_x} \]
where
\[
A = 2 \left( \frac{a}{e^{aT} - e^{aT_0}} \right)^2 \frac{e^{(2(r+a)+\sigma^2)T} - e^{(2(r+a)+\sigma^2)T_0}}{(2 (r + a) + \sigma^2) (r + a + \sigma^2)} - \frac{e^{(r+a)T} e^{(r+a+\sigma^2)T_0} - e^{(2(r+a)+\sigma^2)T_0}}{(r + a) (r + a + \sigma^2)} \]

Therefore, the approximation formulae for pricing of arithmetic weighted average are described below,
\[
C_{AWA} = S(T_0) e^{(r_x - r)(T - T_0)} N \left( d + \sigma_x \sqrt{T - T_0} \right) - K e^{-r(T - T_0)} N (d)
\]
\[
P_{AWA} = -S(T_0) e^{(r_x - r)(T - T_0)} N \left( -d - \sigma_x \sqrt{T - T_0} \right) + K e^{-r(T - T_0)} N (-d)
\]
(16)

where
\[
d = \frac{\log (S(T_0) / K) + (r_x - \sigma_x^2 / 2) (T - T_0)}{\sigma_x \sqrt{T - T_0}}
\]
\[
r_x = \frac{1}{T - T_0} \log \left( \frac{a}{e^{aT} - e^{aT_0}} \left( \frac{e^{(r+a)T} - e^{(r+a)T_0}}{r + a} \right) \right)
\]
\[
\sigma_x = \sqrt{\frac{1}{T - T_0} \log A - 2r_x}
\]
\[
A = 2 \left( \frac{a}{e^{aT} - e^{aT_0}} \right)^2 \frac{e^{(2(r+a)+\sigma^2)T} - e^{(2(r+a)+\sigma^2)T_0}}{(2 (r + a) + \sigma^2) (r + a + \sigma^2)} - \frac{e^{(r+a)T} e^{(r+a+\sigma^2)T_0} - e^{(2(r+a)+\sigma^2)T_0}}{(r + a) (r + a + \sigma^2)} \]
The differences of premiums between geometric weighted average and arithmetic weighted average are showed for each case of different strike prices, remaining periods and volatilities. The values of the difference depicted on Figures 4, 5 and 6 are computed by subtracting the premium for geometric weighted average from that for arithmetic weighted average at each value of $a$.

Let $S=100$, $\sigma=20\%$, non-risk rate=0.5\%, and $T=1$ year. The variable $a$ is allowed to vary $-30$ to 30. For exercise prices, $K=95, 100, 105$ the difference of premiums for the two different weighted averages show similar patterns while the value peak at $a=0$. (Fig.4) Also, none of the shape of patterns is symmetrical about $a=0$, having more heavy tail on positive large values of $a$. Holding the parameters, $S, \sigma, r$ being the same as above the exercise price is fixed with $K=100$. The patterns are quite different for remaining periods $T=1$ month, $T=6$ months, $T=12$ months. The shorter the remaining period becomes the lower the difference is observed. (Fig.5)

Different volatilities, $\sigma=10,20,30\%$ show different values of differences. It is noted, in particular, that the value of the difference is more than 0.08 at around $a=0$. 

15
Figure 4: Differences for exercise prices

Figure 5: Differences for remaining periods

Figure 6: Differences for volatilities
6 COMPARISON FOR THREE METHODOLOGIES

In order to compare the methodologies developed for weighted average call option in the previous sections, we set the following numerical example. The weights vary -20 to 20, implying more weights placed on contract date and maturity date. Let $S=100$ with $K=95,100,105$. The premium values for $T=6,12$ and for $\sigma=10\%, 20\%, 30\%$ are presented in Table 1 through Table 3. Simulation was performed 15000 times and observations 60 times.

The standard deviations, as given in parentheses, for the approximation, are very small due to the variance reduction technique. It can be observed that there are much differences of option premiums between for more weights placed on around maturity date and for more weights placed on around contract date. More weights placed on around maturity date lead to higher average values of call option prices, in particular, higher volatilities and longer remaining periods make the values conspicuous.

Though three methodologies for the values of call option premium don’t show noticeable difference, the difference being occurring only in the third decimal, in many cases, the premium values for Levy shows the highest in all methodologies for the case where more weights are placed on contract date. However, for the situation where more weights are placed on maturity date Kemna & Vorst record higher values of premium than those of other ways.
### Table 1: Premiums for call option with volatility, 10%

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$C_A$: the results based on Kemna & Vorst

$C_B$: the results based on Vorst

$C_C$: the results based on Levy

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Table 2: Premiums for call option with volatility, 20%

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Table 3: Premiums for call option with volatility, 30%

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20
7 CONCLUSION

In this study, a method for the valuation of options based on weighted average instead of simple average was proposed, extending the classical results to more general weighted average option. In particular, a control variable to adjust the weights is incorporated in the weights so that the proposed formulae for option premiums may cope with more different situations and phenomena incurred by underlying assets.

We calculated and compared the prices of an average option based on weighted average in two different ways. (i) Monte Carlo simulation with some adjustments for analytical formula of geometric average option. (ii) use of a new variable whose first and second moments correspond up to those of arithmetic average, which makes B-S formulae applicable to use.

As a future work the values of options over a multi-period will be considered for which each value for the period may be observed as usual time series and hence we can analyze the option problems with time series models.

References


