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**NEW GENERALIZED INVEXITY FOR DUALITY
IN MULTIOBJECTIVE PROGRAMMING
PROBLEMS INVOLVING N -SET FUNCTIONS**

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Abstract In this paper, we introduce four types of generalized convexity for an n -set function and discuss optimality and duality for a multiobjective programming problem involving n -set functions. Under some mild assumption on the new generalized convexity, we present a few optimality conditions for an efficient solution and a weakly efficient solution to the

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problem. Also we prove a weak duality theorem and a strong duality theorem for the problem and its Mond-Weir and general Mond-Weir dual problems respectively.

Keywords: multiobjective programming, n -set function, optimality, duality, generalized convexity

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1. Introduction

In this paper, we consider the following multiobjective programming problem involving n -set functions:

$$(VP) \quad \begin{array}{ll} \text{minimize} & F(S) := (F_1(S), \dots, F_p(S)) \\ \text{subject to} & G_j(S) \leq 0, j \in M, \\ & S := (S_1, \dots, S_n) \in \Omega^n, \end{array}$$

where Ω^n is the n -fold product of a σ -algebra Ω of subsets of a given set X , F_i , $i \in P := \{1, 2, \dots, p\}$ and G_j , $j \in M := \{1, 2, \dots, m\}$ are real-valued functions defined on Ω^n . Let $\Omega_0 := \{S : S \in \Omega^n, G(S) \leq 0\}$ be the set of all the feasible solutions to (VP), where $G := (G_1, G_2, \dots, G_m)$.

Much attention has been paid to analysis of optimization problems with set functions, for example see Chou et al. (1985), Chou et al. (1986), Corley (1987), Kim et al. (1998), Lin (1990)-Lin (1992), Morris (1979), Preda (1991); Preda (1995), Preda and Stancu-Minasian (1997); Preda and Stancu-Minasian (1999) and Zalmai (1991). A formulation for optimization problems with set functions was first given by Morris (1979). The main results of Morris (1979) are confined only to set functions of a single set. Corley (1987) gave the concepts of a partial derivative and a derivative of real-valued n -set functions. Chou et al. (1985), Chou et al. (1986), Kim et al. (1998), Lin (1990)-Lin (1992), Preda (1991), Preda (1995), Preda and Stancu-Minasian (1997), and Preda and Stancu-Minasian (1999) studied optimality and duality for optimization problems involving vector-valued n -set functions. For details, one can refer to Bector and Sihgh (1996), Hsia and Lee (1987), Kim et al. (1998), Lin (1990)-Lin (1992), Mazzoleni (1979), Preda (1995), Rosenmuller and Weidner (1974), Tanaka and Maruyama (1984) and Zalmai (1990).

Starting from the methods used by Jeyakumar and Mond (1992), and Ye (1991), Preda and Stancu-Minasian (2001) defined some new classes of scalar and vector functions called d -type-I, d -pseudo type-I and d -quasi type-I for a multiobjective programming problem involving n -set

functions and obtained a few interesting results on optimality and the Wolfe duality .

Recently, Aghezzaf and Hachimi (2000) introduced new classes of generalized type-I vector valued functions which are different from those defined in Kaul et al. (1994). For details, see Aghezzaf and Hachimi (2000). In this paper, we extend generalized type-I vector valued functions in Aghezzaf and Hachimi (2000) to n -set functions and establish optimality and the Mond–Weir type and general Mond–Weir type duality results for the problem (VP).

2. Definitions and Preliminaries

In this section, we introduce some notions and definitions. For $x := (x_1, x_2, \dots, x_m)$ and $y := (y_1, y_2, \dots, y_m) \in R^m$, we denote $x \leq y$ iff $x_i \leq y_i$ for each $i \in M$; $x \leq y$ iff $x_i \leq y_i$ for each $i \in M$ with $x \neq y$; $x < y$ iff $x_i < y_i$ for each $i \in M$ and $x \not\leq y$ is the negation of $x < y$. We note that $x \in R_+^m$ iff $x \geq 0$.

Let (X, Ω, μ) be a finite atomless measure space with $L_1(X, \Omega, \mu)$ separable, and let d be the pseudo metric on Ω^n defined by $d(S, T) := \left[\sum_{k=1}^n \mu^2(S_k \Delta T_k) \right]^{1/2}$, where $S = (S_1, \dots, S_n)$ and $T := (T_1, \dots, T_n) \in \Omega^n$ and Δ denotes the symmetric difference. Thus, (Ω^n, d) is a pseudometric space which will serve as the domain for most of the functions in this paper.

For $h \in L_1(X, \Omega, \mu)$ and $S \in \Omega_0$ with the indicator (characteristic) function $I_S \in L_\infty(X, \Omega, \mu)$, the integral $\int h d\mu$ is denoted by $\langle h, I_S \rangle$.

The notion of differentiability for a real-valued set function was originally introduced by Morris (1979); and its n -set counterpart was discussed in Corley (1987).

A function $\varphi : \Omega \rightarrow R$ is differentiable at $S \in \Omega$ if there exist $D\varphi(S^0) \in L_1(X, S, \mu)$ called the derivative of f at $S^0 := (S_1^0, S_2^0, \dots, S_n^0)$, and $\psi : \Omega \times \Omega \rightarrow R$ such that, for each $S \in \Omega$

$$\varphi(S) = \pi(S^0) + \langle D\varphi(S^0), I_S - I_{S^0} \rangle + \psi(S, S^0),$$

where $\psi(S, S^0)$ is $o(d(S, S^0))$, that is, $\lim_{d(S, S^0) \rightarrow 0} \frac{\psi(S, S^0)}{d(S, S^0)} = 0$.

Function $F : \Omega^n \rightarrow R$ is said to have a partial derivative at S^0 with respect to its k -th argument if the function $\varphi(S_k) = F(S_1^0, \dots, S_{k-1}^0, S_k^0, S_{k+1}^0, \dots, S_n^0)$ has derivative $D\varphi(S_k^0)$. We define

$$D_k F(S^0) := D\varphi(S_k^0)$$

and denote

$$DF(S^0) := (D_1F(S^0), \dots, D_nF(S^0)).$$

Function $F : \Omega^n \rightarrow R$ is said to be differentiable at S^0 if there exist $DF(S^0)$ and $\psi : \Omega^n \times \Omega^n \rightarrow R$ such that

$$F(S) = F(S^0) + \sum_{k=1}^n \left\langle D_kF(S^0), I_{S_k} - I_{S_k^0} \right\rangle + \psi(S, S^0),$$

where $\psi(S, S^0)$ is $o(d(S, S^0))$.

A feasible solution S^0 of (VP) is said to be an efficient solution of (VP) if there exists no other feasible solution S of (VP) such that $F_i(S) \leq F_i(S^0)$, for all $i \in P$, with strict inequality for at least one $i \in P$.

A feasible solution S^0 of (VP) is said to be a weakly efficient solution of (VP) if there exists no other feasible solution S of (VP) such that $F_i(S) < F_i(S^0)$, for all $i \in P$.

Along the lines of Jeyakumar and Mond (1992) and Aghezzaf and Hachimi (2000), we define the following types of n -set functions, called d -strong pseudoquasi-type-I, d -weak strictly-pseudo quasi-type-I, d -weak strictly pseudo-type-I, d -weak quasi strictly-pseudo-type-I functions.

Definition 2.1 (F, G) is said to be d -weak strictly-pseudo quasi-type-I at $S^0 \in \Omega^n$ with respect to $\eta : \Omega^n \times \Omega^n \rightarrow R^n$, $\gamma_i : \Omega^n \times \Omega^n \rightarrow R^+ \setminus \{0\}$, $i = 1, 2, \dots, p$ and $\delta_j : \Omega^n \times \Omega^n \rightarrow R^+ \setminus \{0\}$, $j = 1, \dots, m$, if for every $S \in \Omega_0$

$$\begin{aligned} \sum_{i=1}^p \gamma_i(S, S^0) F_i(S) &\leq \sum_{i=1}^p \gamma_i(S, S^0) F_i(S^0) \\ \Rightarrow \sum_{i=1}^p \sum_{k=1}^p \eta_k(S, S^0) \left\langle D_k F_i(S^0), I_{S_k} - I_{S_k^0} \right\rangle &< 0 \end{aligned}$$

and

$$-\sum_{j=1}^m \delta_j(S, S^0) G_j(S^0) \leq 0 \Rightarrow \sum_{i=1}^p \sum_{k=1}^p \eta_k(S, S^0) \left\langle D_k G_j(S^0), I_{S_k} - I_{S_k^0} \right\rangle \leq 0.$$

It is an extension of weak strictly-pseudo quasi-type-I functions defined in Aghezzaf and Hachimi (2000). The concept also extends the d -pseudo-quasi-type-I functions defined in Preda and Stancu-Minasian (2001). There exist non n -set functions which are weak strict pseudoquasi-type-I, but not strict pseudoquasi-type-I and not type-I with respect to the same η , see Example 2.1 in Aghezzaf and Hachimi (2000).

Definition 2.2 (F, G) is said to be d -strong-pseudo quasi-type-I at $S^0 \in \Omega^n$ with respect to $\eta : \Omega^n \times \Omega^n \rightarrow R^n$, $\gamma_i : S^n \times S^n \rightarrow R^+ \setminus \{0\}$, $i = 1, 2, \dots, p$ and $\delta_j : \Omega^n \times \Omega^n \rightarrow R^+ \setminus \{0\}$, $j = 1, \dots, m$, if for every $S \in \Omega_0$

$$\begin{aligned} \sum_{i=1}^p \gamma_i(S, S^0) F_i(S) &\leq \sum_{i=1}^p \gamma_i(S, S^0) F_i(S^0) \\ \Rightarrow \sum_{i=1}^p \sum_{k=1}^p \eta_k(S, S^0) \langle D_k F_i(S^0), I_{S_k} - I_{S_k^0} \rangle &\leq 0 \end{aligned}$$

and

$$-\sum_{j=1}^m \delta_j(S, S^0) G_j(S^0) \leq 0 \Rightarrow \sum_{i=1}^p \sum_{k=1}^p \eta_k(S, S^0) \langle D_k G_j(S^0), I_{S_k} - I_{S_k^0} \rangle \leq 0.$$

Definition 2.3 (F, G) is said to be d -weak quasi strictly-pseudo-type-I at $S^0 \in \Omega^n$ with respect to $\eta : \Omega^n \times \Omega^n \rightarrow R^n$, $\gamma_i : \Omega^n \times \Omega^n \rightarrow R^+ \setminus \{0\}$, $i = 1, 2, \dots, p$ and $\delta_j : \Omega^n \times \Omega^n \rightarrow R^+ \setminus \{0\}$, $j = 1, \dots, m$, if for every $S \in \Omega_0$

$$\begin{aligned} \sum_{i=1}^p \gamma_i(S, S^0) F_i(S) &\leq \sum_{i=1}^p \gamma_i(S, S^0) F_i(S^0) \\ \Rightarrow \sum_{i=1}^p \sum_{k=1}^p \eta_k(S, S^0) \langle D_k F_i(S^0), I_{S_k} - I_{S_k^0} \rangle &\leq 0 \end{aligned}$$

and

$$-\sum_{j=1}^m \delta_j(S, S^0) G_j(S^0) \leq 0 \Rightarrow \sum_{i=1}^p \sum_{k=1}^p \eta_k(S, S^0) \langle D_k G_j(S^0), I_{S_k} - I_{S_k^0} \rangle \leq 0.$$

Definition 2.4 (F, G) is said to be d -weak strictly pseudo-type-I at $S^0 \in \Omega^n$ with respect to $\eta : \Omega^n \times \Omega^n \rightarrow R^n$, $\gamma_i : \Omega^n \times \Omega^n \rightarrow R^+ \setminus \{0\}$, $i = 1, 2, \dots, p$ and $\delta_j : \Omega^n \times \Omega^n \rightarrow R^+ \setminus \{0\}$, $j = 1, \dots, m$, if for every $S \in \Omega_0$

$$\begin{aligned} \sum_{i=1}^p \gamma_i(S, S^0) F_i(S^0) \\ \Rightarrow \sum_{i=1}^p \sum_{k=1}^p \eta_k(S, S^0) \langle D_k F_i(S^0), I_{S_k} - I_{S_k^0} \rangle < 0 \end{aligned}$$

and

$$-\sum_{j=1}^m \delta_j(S, S^0) G_j(S^0) \leq 0 \Rightarrow \sum_{i=1}^p \sum_{k=1}^p \eta_k(S, S^0) \langle D_k G_j(S^0), I_{S_k} - I_{S_k^0} \rangle < 0.$$

Remark 2.1 *The above definitions are extensions of the corresponding definitions in Aghezzaf and Hachimi (2000). These definitions are different from the other definitions such as in Kaul et al. (1994) and Hanson and Mond (1987), for various examples refer to Aghezzaf and Hachimi (2000).*

The following results from Zalmai (1991) will be needed in Section 4.

Lemma 2.1 *Let S^0 be an efficient (or weakly efficient) solution for (VP) and let $F_i, i \in P$ and $G_j, j \in M$ be differentiable at S^0 . Then there exist $\lambda \in R_+^p, \mu \in R_+^m, (\lambda, \mu) \neq 0$ such that*

$$\sum_{k=1}^n \left\langle \sum_{i=1}^p \lambda_i D_k F_i(S^0) + \sum_{j=1}^m \mu_j D_k G_j(S^0), I_{S_k} - I_{S_k^0} \right\rangle \geq 0 \text{ for all } S \in \Omega^n,$$

$$\mu_j G_j(S^0) = 0, j \in M.$$

Definition 2.5 *A feasible solution S^0 is said to be a regular feasible solution if there exists $\hat{S} \in \Omega^n$ such that*

$$G_j(S^0) + \langle DG_j(S^0), I_{\hat{S}} - I_{S^0} \rangle < 0, j \in M_0.$$

Thus incorporating the above in Lemma 2.1, normalizing λ such that $\sum_{i=1}^p \lambda_i = 1$ and redefining (μ_1, \dots, μ_m) , we can have the following result.

Lemma 2.2 *Let S^0 be an efficient (or weakly efficient) solution for (VP) and let $F_i, i \in P$ and $G_j, j \in M$ be differentiable at S^0 . Then there exist $\lambda \in R_+^p$ with $\sum_{i=1}^p \lambda_i = 1$ and $\mu \in R_+^m$ such that*

$$\sum_{k=1}^n \left\langle \sum_{i=1}^p \lambda_i D_k F_i(S^0) + \sum_{j=1}^m \mu_j D_k G_j(S^0), I_{S_k} - I_{S_k^0} \right\rangle \geq 0, \text{ for all } S \in \Omega^n,$$

$$\mu_j G_j(S^0) = 0, j \in M.$$

3. Optimality Condition

In this section, we give a sufficient optimality condition for a weakly efficient solution to (VP) under the assumption of new types of generalized convexity introduced in Section 2.

Theorem 3.1 *Let S^0 be a feasible solution for (VP). Suppose that*

- (i1) *There exist $\lambda_i^0 \geq 0, i \in P$, with $\sum_{i=1}^p \lambda_i^0 = 1$ and $\mu_j^0 \geq 0, j \in M_0 := M(S^0) := \{j \in M : G_j(S^0) = 0\}$, such that for all $S \in \Omega^n$*

$$\left\langle D_k(\lambda^0 F)(S^0) + D_k(\mu^0 G)(S^0), I_{S_k} - I_{S_k^0} \right\rangle \geq 0;$$

- (i2) *$(F, \mu G)$ is d -strong pseudo quasi-type-I at S^0 with respect to η, γ_i, δ_j and $\lambda > 0$;*
 (i3) *$(F, \mu G)$ is d -weak strictly pseudo quasi-type-I at S^0 with respect to η, γ_i , and δ_j ;*
 (i4) *$(F, \mu G)$ is d -weak strictly pseudo-type-I at S^0 with respect to η, γ_i , and δ_j ;*

with η satisfying $\eta^\top \alpha < 0 \Rightarrow \alpha_k < 0$ for at least one k in $\{1, 2, \dots, n\}$. Then S^0 is a weakly efficient solution to (VP).

Proof. Assume that S^0 is not a weakly efficient solution to (VP). Then there is a feasible solution to (VP) such that

$$F_i(S) < F_i(S^0) \text{ for any } i \in P. \quad (1.1)$$

According to (i2) there exist $\gamma_i : \Omega^n \times \Omega^n \rightarrow R^+ \setminus \{0\}, i = 1, 2, \dots, p$ and $\delta_j : \Omega^n \times \Omega^n \rightarrow R^+ \setminus \{0\}, j \in M_0$ such that, for all $S \in \Omega_0$

$$\begin{aligned} \sum_{i=1}^p \gamma_i(S, S^0) \lambda_i^0 F_i(S) &< \sum_{i=1}^p \gamma_i(S, S^0) \lambda_i^0 F_i(S^0) \\ \Rightarrow \sum_{i=1}^p \sum_{k=1}^n \lambda_i^0 \eta_k(S, S^0) \left\langle D_k F_i(S^0), I_{S_k} - I_{S_k^0} \right\rangle &< 0 \text{ (since } \lambda > 0) \end{aligned} \quad (1.2)$$

and

$$\begin{aligned} - \sum_{j \in M_0} \delta_j(S, S^0) \mu_j^0 G_j(S^0) &\leq 0 \\ \Rightarrow \sum_{j \in M_0} \sum_{k=1}^n \mu_j^0 \eta_k(S, S^0) \left\langle D_k G_j(S^0), I_{S_k} - I_{S_k^0} \right\rangle &\leq 0. \end{aligned} \quad (1.3)$$

From (1.1) and $\lambda_i^0 \geq 0, i \in P$, with $\sum_{i=1}^p \lambda_i^0 = 1$ and $\gamma_i > 0$ for any $i \in P$, we get

$$\sum_{i=1}^p \gamma_i(S, S^0) \lambda_i^0 F_i(S) < \sum_{i=1}^p \gamma_i(S, S^0) \lambda_i^0 F_i(S^0).$$

Using (1.2), we get

$$\begin{aligned} & \sum_{i=1}^p \sum_{k=1}^n \lambda_i^0 \eta_k(S, S^0) \langle D_k F_i(S^0), I_{S_i} - I_{S_i^0} \rangle < 0 \\ \Rightarrow & \sum_{k=1}^n \eta_k(S, S^0) \langle D_k F_i(S^0), I_{S_k} - I_{S_k^0} \rangle < 0. \end{aligned} \quad (1.4)$$

Since S^0 is a feasible solution to (VP) and $G_j(S^0) = 0$ for $j \in M_0$, we obtain

$$- \sum_{j \in M_0} \delta_j(S, S^0) \mu_j^0 G_j(S^0) \leq 0.$$

This relation together with (1.3) implies

$$\begin{aligned} & \sum_{j \in M_0} \sum_{k=1}^n \mu_j^0 \eta_k(S, S^0) \langle D_k G_j(S^0), I_{S_k} - I_{S_k^0} \rangle \leq 0 \\ \Rightarrow & \sum_{k=1}^n \eta_k(S, S^0) \langle D_k(\mu^0 G)(S^0), I_{S_k} - I_{S_k^0} \rangle \leq 0. \end{aligned} \quad (1.5)$$

By (1.4) and (1.5), we get

$$\begin{aligned} & \sum_{k=1}^n \eta_k(S, S^0) \langle D_k(\lambda^0 F)(S^0) + D_k(\mu^0 G)(S^0), I_{S_k} - I_{S_k^0} \rangle < 0 \\ \Rightarrow & \langle D_k(\lambda^0 F)(S^0) + D_k(\mu^0 G)(S^0), I_{S_k} - I_{S_k^0} \rangle < 0, \end{aligned}$$

for at least one k in $\{1, 2, \dots, n\}$ (because $\eta^\top \alpha < 0 \Rightarrow \alpha_k < 0$ for at least one k in $\{1, 2, \dots, n\}$), which contradicts (i1).

By (i3), there exist $\gamma_i : \Omega^n \times \Omega^n \rightarrow R^+ \setminus \{0\}, i = 1, 2, \dots, p$ and $\delta_j : \Omega^n \times \Omega^n \rightarrow R^+ \setminus \{0\}, j \in M_0$ such that, for all $S \in \Omega_0$

$$\begin{aligned} & \sum_{i=1}^p \gamma_i(S, S^0) \lambda_i^0 F_i(S) \leq \sum_{i=1}^p \gamma_i(S, S^0) \lambda_i^0 F_i(S^0) \\ \Rightarrow & \sum_{i=1}^p \sum_{k=1}^n \lambda_i^0 \eta_k(S, S^0) \langle D_k F_i(S^0), I_{S_k} - I_{S_k^0} \rangle < 0 \end{aligned} \quad (1.6)$$

and

$$\begin{aligned}
 & - \sum_{j \in M_0} \delta_j(S, S^0) \mu_j^0 G_j(S^0) \leq 0 \\
 & \Rightarrow \sum_{j \in M_0} \sum_{k=1}^n \mu_j^0 \eta_k(S, S^0) \left\langle D_k G_j(S^0), I_{S_k} - I_{S_k^0} \right\rangle \leq 0.
 \end{aligned} \tag{1.7}$$

By (1.1) and $\lambda_i^0 \geq 0$, $i \in P$, with $\sum_{i=1}^p \lambda_i^0 = 1$ and $\gamma_i > 0$ for any $i \in P$, we get

$$\sum_{i=1}^p \gamma_i(S, S^0) \lambda_i^0 F_i(S) \leq \sum_{i=1}^p \gamma_i(S, S^0) \lambda_i^0 F_i(S^0).$$

Using (1.6) and the above inequality, we get

$$\begin{aligned}
 & \sum_{i=1}^p \sum_{k=1}^n \lambda_i^0 \eta_k(S, S^0) \left\langle D_k F_i(S^0), I_{S_k} - I_{S_k^0} \right\rangle < 0 \\
 & \Rightarrow \sum_{k=1}^n \eta_k(S, S^0) \left\langle D_k(\lambda^0 F)(S^0), I_{S_k} - I_{S_k^0} \right\rangle < 0.
 \end{aligned} \tag{1.8}$$

From the feasibility of S and (1.7), we get (1.5). By (1.8) and (1.5), we get

$$\left\langle D_k(\lambda^0 F)(S^0) + D_k(\mu^0 G)(S^0), I_{S_k} - I_{S_k^0} \right\rangle < 0,$$

for at least one k in $\{1, 2, \dots, n\}$ (because $\eta^\top \alpha < 0 \Rightarrow \alpha_k < 0$ for at least one k in $\{1, 2, \dots, n\}$), which again contradicts (i1).

By (i4), there exist $\gamma_i : \Omega^n \times \Omega^n \rightarrow R^+ \setminus \{0\}$, $i = 1, 2, \dots, p$ and $\delta_j : \Omega^n \times \Omega^n \rightarrow R^+ \setminus \{0\}$, $j \in M_0$ such that, for all $S \in \Omega_0$, we get (1.6) and

$$\begin{aligned}
 & - \sum_{j \in M_0} \delta_j(S, S^0) \mu_j^0 G_j(S^0) \leq 0 \\
 & \Rightarrow \sum_{j=1}^m \sum_{k=1}^n \eta_k(S, S^0) \left\langle D_k G_j(S^0), I_{S_k} - I_{S_k^0} \right\rangle < 0.
 \end{aligned} \tag{1.9}$$

By (1.1) and $\lambda_i^0 \geq 0$, $i \in P$, with $\sum_{i=1}^p \lambda_i^0 = 1$ and $\gamma_i > 0$ for any $i \in P$, we get

$$\sum_{i=1}^p \gamma_i(S, S^0) \lambda_i^0 F_i(S) \leq \sum_{i=1}^p \gamma_i(S, S^0) \lambda_i^0 F_i(S^0).$$

Using (1.6) and the above inequality, we get (1.8). From the feasibility of S and (1.9), we get (1.5). By (1.8) and (1.5), we have

$$\left\langle D_k(\lambda^0 F)(S^0) + D_k(\mu^0 G)(S^0), I_{S_k} - I_{S_k^0} \right\rangle < 0,$$

for at least one k in $\{1, 2, \dots, n\}$ (because $\eta^\top \alpha < 0 \Rightarrow \alpha_k < 0$ for at least one k in $\{1, 2, \dots, n\}$), which contradicts (i1). This completes the proof. \square

4. Mond-Weir Duality

In this section, we consider the following Mond-Weir dual problem(MD):

$$\begin{aligned}
& \text{maximize} && F(T) \\
& \text{subject to} && \left\langle D_k(\lambda F)(T) + D_k(\mu G)(T), I_{S_k} - I_{T_k} \right\rangle \geq 0, \text{ for all } S \in \Omega^n, \\
& && \sum_{j=1}^m \mu_j G_j(T) \geq 0, \\
& && \lambda_i \geq 0, i \in P, \text{ and } \sum_{i=1}^p \lambda_i = 1, \\
& && \mu_j \geq 0, j \in M \text{ and } T \in \Omega^n.
\end{aligned} \tag{1.10}$$

Let D be the set of all feasible solutions to (MD).

Theorem 4.1 (*Weak Duality*). *Suppose that $S \in \Omega_0$ and $(T, \lambda, \mu) \in D$. If any one of the following conditions is satisfied:*

- (a) $(F, \mu G)$ is d -strong-pseudo quasi-type-I at T with respect to η , γ_i , δ_j and $\lambda > 0$;
- (b) $(F, \mu G)$ is d -weak strictly pseudo quasi-type-I at T with respect to η , γ_i and δ_j ;
- (c) $(F, \mu G)$ is d -weak strictly pseudo-type-I at T with respect to η , γ_i and δ_j ;

with η satisfying $\eta^\top \alpha < 0 \Rightarrow \alpha_k < 0$ for at least one k in $\{1, 2, \dots, n\}$, then $F(S) \geq F(T)$.

Proof. We proceed by contradiction. If there exist $S \in \Omega_0$ and $(T, \lambda, \mu) \in D$ such that $F(S) < F(T)$. Since γ_i is in $R^+ \setminus \{0\}$, we have

$$\sum_{i=1}^p \gamma_i(S, T) F_i(S) \leq \sum_{i=1}^p \gamma_i(S, T) F_i(T). \tag{1.11}$$

Since $(T, \lambda, \mu) \in D$, it follows that

$$\sum_{j=1}^m \mu_j G_j(T) \geq 0, \text{ that is, } - \sum_{j=1}^m \mu_j G_j(T) \leq 0$$

Because δ_j is in $R^+ \setminus \{0\}$, we have

$$-\sum_{j=1}^m \delta_j(S, T) \mu_j G_j(T) \leq 0. \quad (1.12)$$

By condition (a), (1.11) and (1.12) yield

$$\sum_{i=1}^p \sum_{k=1}^n \lambda_i \eta_k(S, T) \langle D_k F_i(T), I_{S_k} - I_{T_k} \rangle \leq 0$$

and

$$\sum_{j=1}^m \sum_{k=1}^n \mu_j \eta_k(S, T) \langle D_k G_j(T), I_{S_k} - I_{T_k} \rangle \leq 0.$$

Since $\lambda > 0$, the above two inequalities imply

$$\sum_{k=1}^n \eta_k(S, T) \langle D_k(\lambda F)(T), I_{S_k} - I_{T_k} \rangle < 0$$

and

$$\sum_{k=1}^n \eta_k(S, T) \langle D_k(\mu G)(T), I_{S_k} - I_{T_k} \rangle \leq 0.$$

By the above two inequalities, we get

$$\sum_{k=1}^n \eta_k(S, T) \langle D_k(\lambda F)(T) + D_k(\mu G)(T), I_{S_k} - I_{T_k} \rangle < 0,$$

$$\Rightarrow \langle D_k(\lambda F)(T) + D_k(\mu G)(T), I_{S_k} - I_{T_k} \rangle < 0,$$

for at least one k in $\{1, 2, \dots, n\}$ (because $\eta^\top \alpha < 0 \Rightarrow \alpha_k < 0$ for at least one k in $\{1, 2, \dots, n\}$), which contradicts (1.10).

By condition (b), we get

$$\sum_{i=1}^p \sum_{k=1}^n \lambda_i \eta_k(S, T) \langle D_k F_i(T), I_{S_k} - I_{T_k} \rangle < 0$$

and

$$\sum_{j=1}^m \sum_{k=1}^n \mu_j \eta_k(S, T) \langle D_k G_j(T), I_{S_k} - I_{T_k} \rangle \leq 0.$$

These two inequalities imply

$$\sum_{k=1}^n \eta_k(S, T) \langle D_k(\lambda F)(T), I_{S_k} - I_{T_k} \rangle < 0$$

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and

$$\sum_{k=1}^n \eta_k(S, T) \langle D_k(\mu G)(T), I_{S_k} - I_{T_k} \rangle \leq 0.$$

By these two inequalities, we get

$$\begin{aligned} & \sum_{k=1}^n \eta_k(S, T) \langle D_k(\lambda F)(T) + D_k(\mu G)(T), I_{S_k} - I_{T_k} \rangle < 0 \\ & \Rightarrow \langle D_k(\lambda F)(T) + D_k(\mu G)(T), I_{S_k} - I_{T_k} \rangle < 0, \end{aligned}$$

for at least one k in $\{1, 2, \dots, n\}$ (because $\eta^\top \alpha < 0 \Rightarrow \alpha_k < 0$ for at least one k in $\{1, 2, \dots, n\}$). This contradicts (1.10).

By condition (c), (1.11) and (1.12) imply

$$\sum_{i=1}^p \sum_{k=1}^n \lambda_i \eta_k(S, T) \langle D_k F_i(T), I_{S_k} - I_{T_k} \rangle < 0$$

and

$$\sum_{j=1}^m \sum_{k=1}^n \mu_j \eta_k(S, T) \langle D_k G_j(T), I_{S_k} - I_{T_k} \rangle < 0.$$

These two inequalities imply

$$\sum_{k=1}^n \eta_k(S, T) \langle D_k(\lambda F)(T), I_{S_k} - I_{T_k} \rangle < 0$$

and

$$\sum_{k=1}^n \eta_k(S, T) \langle D_k(\mu G)(T), I_{S_k} - I_{T_k} \rangle < 0.$$

By these two inequalities, we get

$$\begin{aligned} & \sum_{k=1}^n \eta_k(S, T) \langle D_k(\lambda F)(T) + D_k(\mu G)(T), I_{S_k} - I_{T_k} \rangle < 0, \\ & \Rightarrow \langle D_k(\lambda F)(T) + D_k(\mu G)(T), I_{S_k} - I_{T_k} \rangle < 0, \end{aligned}$$

for at least one k in $\{1, 2, \dots, n\}$ (because $\eta^\top \alpha < 0 \Rightarrow \alpha_k < 0$ for at least one k in $\{1, 2, \dots, n\}$), which contradicts (1.10). This completes the proof. \square

Theorem 4.2 (Strong Duality). *Let $S^0 \in \Omega_0$ be such that:*
(b1) S^0 is a weakly efficient solution to (VP),

(b2) S^0 is a regular solution to (VP).

Then there exist $\lambda^0 \in R^p$ and $\mu^0 \in R^m$ such that (S^0, λ^0, μ^0) is a feasible solution for (MD) and the values of the objective functions of (VP) and (MD) are equal at these points. Furthermore, if the conditions of the weak duality in Theorem 4.1 hold for each feasible solution (T, λ, μ) of (MD), then (S^0, λ^0, μ^0) is a weakly efficient solution to (MD).

Proof. By Lemma 2.1, there exist $\lambda_i^0 \geq 0, i \in P$ with $\sum_{i=1}^p \lambda_i^0 = 1$ and $\mu_j^0 \geq 0, j \in M_0$ such that (S^0, λ^0, μ^0) is feasible for (MD) and the values of the objective functions of (VP) and (MD) are equal. The last part follows directly from Theorem 4.1. \square

5. General Mond-Weir Duality

In this section, we study a general type of Mond-Weir duality and establish weak and strong duality theorems under a generalized invexity assumption.

Consider the following general Mond-Weir type of dual problem (GMD):

$$\begin{aligned} & \text{maximize} && F(T) + \sum_{j \in J_0} \mu_j G_j(T) \\ & \text{subject to} && \left\langle D_k(\lambda F)(T) + D_k(\mu G)(T), I_{S_k} - I_{T_k} \right\rangle \geq 0, \end{aligned} \quad (1.13)$$

$$\sum_{j \in J_\alpha} \mu_j G_j(T) \geq 0 \text{ for } 1 \leq \alpha \leq r, \quad (1.14)$$

$$\lambda \geq 0, \mu \geq 0 \text{ and } \sum_{i=1}^p \lambda_i = 1,$$

where $J_\alpha, 0 \leq \alpha \leq r$ are partitions of set M .

Theorem 5.1 (*Weak Duality*). *Suppose if for all $S \in \Omega_0$ and $(T, \lambda, \mu) \in D$, the following conditions hold:*

- (a) $\lambda > 0$ and $\left((F + \sum_{j \in J_0} \mu_j G_j)(\cdot), \sum_{j \in J_\alpha} \mu_j G_j(\cdot) \right)$ is d -strong pseudo quasi-type-I at T with respect to η, γ_i , and δ_j for any $\alpha, 1 \leq \alpha \leq r$;
- (b) $\left((F + \sum_{j \in J_0} \mu_j G_j)(\cdot), \sum_{j \in J_\alpha} \mu_j G_j(\cdot) \right)$ is d -weak strictly pseudo quasi-type-I at T with respect to η, γ_i , and δ_j for any $\alpha, 1 \leq \alpha \leq r$;
- (c) $\left((F + \sum_{j \in J_0} \mu_j G_j)(\cdot), \sum_{j \in J_\alpha} \mu_j G_j(\cdot) \right)$ is d -weak strictly pseudo-type-I at T with respect to η, γ_i , and δ_j for any $\alpha, 1 \leq \alpha \leq r$;

with η satisfying $\eta^\top \alpha < 0 \Rightarrow \alpha_k < 0$ for at least one k in $\{1, 2, \dots, n\}$, then the following can not hold:

$$F(S) \leq F(T) + \sum_{j \in J_0} \mu_j G_j(T).$$

Proof. Suppose to the contrary that the result holds. Since $S \in \Omega_0$ and $\mu \geq 0$, we have

$$F(S) + \sum_{j \in J_0} \mu_j G_j(S) \leq F(T) + \sum_{j \in J_0} \mu_j G_j(T).$$

From (1.14), we have

$$- \sum_{j \in J_\alpha} \mu_j G_j(T) \leq 0, \text{ for all } 1 \leq \alpha \leq r.$$

Since γ_i and δ_j are in $R^+ \setminus \{0\}$ from the above two inequalities, we have

$$\begin{aligned} & \sum_{i=1}^p \gamma_i(S, T) \left(F_i(S) + \sum_{j \in J_0} \mu_j G_j(S) \right) \\ & \leq \sum_{i=1}^p \gamma_i(S, T) \left(F_i(T) + \sum_{j \in J_0} \mu_j G_j(T) \right) \end{aligned} \quad (1.15)$$

and

$$- \sum_{j \in J_\alpha} \delta_j(S, T) \mu_j G_j(T) \leq 0. \quad (1.16)$$

By condition (a), (1.15) and (1.16), we have

$$\sum_{i=1}^p \sum_{k=1}^n \eta_k(S, T) \left\langle D_k \left(F_i(T) + \sum_{j \in J_0} \mu_j G_j(T) \right), I_{S_k} - I_{T_k} \right\rangle \leq 0 \quad (1.17)$$

and

$$\sum_{j \in J_\alpha} \sum_{k=1}^n \mu_j \eta_k(S, T) \langle D_k G_j(T), I_{S_k} - I_{T_k} \rangle \leq 0, 1 \leq \alpha \leq r. \quad (1.18)$$

Since $\lambda > 0$, from (1.17) and (1.18), we have

$$\sum_{i=1}^p \eta_k(S, T) \left\langle D_k \left((\lambda F)(T) + \sum_{\alpha=0}^r (\mu_{J_\alpha} G_{J_\alpha})(T) \right), I_{S_k} - I_{T_k} \right\rangle < 0. \quad (1.19)$$

Since J_0, J_1, \dots, J_r are partitions of M , (1.19) is equivalent to

$$\begin{aligned} & \sum_{k=1}^n \eta_k(S, T) \langle D_k(\lambda F)(T) + D_k(\mu G)(T), I_{S_k} - I_{T_k} \rangle < 0. \\ & \Rightarrow \langle D_k(\lambda F)(T) + D_k(\mu G)(T), I_{S_k} - I_{T_k} \rangle < 0, \end{aligned} \quad (1.20)$$

for at least one one k in $\{1, 2, \dots, n\}$ (because $\eta^\top \alpha < 0 \Rightarrow \alpha_k < 0$ for at least one k in $\{1, 2, \dots, n\}$), which contradicts (1.13).

Using condition (b), from (1.15) and (1.16), we get

$$\sum_{i=1}^p \sum_{k=1}^n \eta_k(S, T) \left\langle D_k \left(F_i(T) + \sum_{j \in J_0} \mu_j G_j(T) \right), I_{S_k} - I_{T_k} \right\rangle < 0$$

and

$$\sum_{j \in J_\alpha} \sum_{k=1}^n \mu_j \eta_k(S, T) \langle D_k G_j(T), I_{S_k} - I_{T_k} \rangle \leq 0, 1 \leq \alpha \leq r.$$

Since $\lambda \geq 0$, the above inequalities give (1.20) and then again we get a contradiction to (1.13).

Suppose now that (c) is satisfied. From (1.15) and (1.16) it follows that

$$\sum_{i=1}^p \sum_{k=1}^n \eta_k(S, T) \left\langle D_k \left(F_i(T) + \sum_{j \in J_0} \mu_j G_j(T) \right), I_{S_k} - I_{T_k} \right\rangle < 0$$

and

$$\sum_{j \in J_\alpha} \sum_{k=1}^n \mu_j \eta_k(S, T) \langle D_k G_j(T), I_{S_k} - I_{T_k} \rangle < 0, 1 \leq \alpha \leq r.$$

Since $\lambda \geq 0$, the above inequalities give (1.20) and then again we get a contradiction to (1.13). This completes the proof. \square

Theorem 5.2 (*Strong Duality*). *Let $S^0 \in \Omega_0$ satisfy that:*

(b1) S^0 is a weakly efficient solution to (VP);

(b2) S^0 is a regular solution to (VP).

Then there exist $\lambda^0 \in R^p$ and $\mu^0 \in R^m$ such that (S^0, λ^0, μ^0) is a feasible solution for (GMD) and $\mu_{J_0} G_{J_0}(S^0) = 0$, and the values of the objective functions of (VP) and (GMD) at these solutions are equal. Furthermore, if the weak duality holds between (VP) and (GMD), then (S^0, λ^0, μ^0) is a weakly efficient solution to (GMD).

The proof of this theorem follows the lines of the proof of Theorem 4.2 in the light of Theorem 5.1.

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