MS-02-07

Fractional Programming: the sum-of-ratios case

Siegfried Schaible and Jianming Shi

December 02, 2002

Fractional Programming: the sum-of-ratios case *

Siegfried Schaible† and Jianming Shi‡

November, 2002

Abstract

One of the most difficult fractional programs encountered so far is the sum-of-ratios problem. Contrary to earlier expectations it is much more removed from convex programming than other multi-ratio problems analyzed before. It really should be viewed in the context of global optimization. It proves to be *essentially* $\mathcal{NP}$-complete in spite of its special structure under the usual assumptions on numerators and denominators. The paper provides a recent survey of applications, theoretical results and various algorithmic approaches for this challenging problem.

Keywords: fractional programming, multi-ratio problems, sum-of-ratios, branch-and-bound.

AMS Classification 2000: 90C32

1 Introduction

The problem of optimizing one or several ratios of functions is called a *fractional program*. In a comprehensive bibliography [50] by the first author in the *Handbook of Global Optimization* [28] well over one thousand contributions to fractional programming are listed. Similarly the recent monograph by Stancu-Minasian [60] on fractional programming contains almost as many entries. Meanwhile the literature on this class of nonlinear programs has kept growing.

Apart from isolated earlier results, most of the work in fractional programming was done since about 1960. The analysis of fractional programs with only one ratio has largely dominated the literature until about 1980. The first monograph [47] in fractional programming published by the first author in 1978 extensively covers applications, theoretical results and algorithms for single-ratio fractional programs; see also [48, 51] .

*This research was partially supported by a Grant-in-Aid (C-14550405) from the Ministry of Education, Science, Sports and Culture of Japan.

†A.G. Anderson Graduate School of Management, University of California Riverside, CA 92521, USA. Email: siegfried.schaible@ucr.edu

‡School of Management, Tokyo University of Science, Saitama 346-8512, JAPAN. Email: shi@ms.kuki.tus.ac.jp

In 1988 a second monograph appeared authored by Craven [17]. It includes some early results on multi-ratio fractional programs too. Since the first international conference with an emphasis on fractional programming, the NATO Advanced Study Institute on *Generalized Convexity in Optimization and Economics* in 1980 (see [53]), a series of similar international conferences was held which demonstrates a shift of interest from the single-ratio to the multi-ratio case [58, 11, 31, 16, 25]. The third monograph on fractional programming was authored by Stancu-Minasian [60] in 1997. It includes various results for the multi-ratio case too.

The present article focuses on the sum-of-ratios fractional program which until about 1990 was one of the least researched fractional programs. For the necessary background on the single-ratio problem we refer to the chapter on fractional programming in the *Handbook of Global Optimization* [50]. In Sections 2, 3 and 4 below we provide a recent survey of applications, theoretical results and algorithmic advances for the sum-of-ratios problem, respectively.

We consider

$$\max \left\{ \sum_{i=1}^{p} \frac{f_i(x)}{g_i(x)} \,\middle|\, x \in S \right\} \tag{1.1}$$

and

$$\min \left\{ \sum_{i=1}^{p} \frac{f_i(x)}{g_i(x)} \,\middle|\, x \in S \right\}. \tag{1.2}$$

We make the following assumptions. The feasible region S is a convex set of R^n. In case of (1.1) $f_i : R^n \to R$ are nonnegative concave and $g_i : R^n \to R$ are positive, convex for each i. In case of (1.2) $f_i : R^n \to R$ are nonnegative convex and $g_i : R^n \to R$ are positive, concave for each i. Often these properties are only required on the feasible region S.

2 Applications

Models (1.1), (1.2) arise naturally in decision making when several rates are to be optimized simultaneously and a compromise is sought which optimizes a weighted sum of these rates. In light of the applications of single-ratio fractional programming (e.g. [48, 51, 50]) numerators and denominators may be representing output, input, profit, cost, capital, risk or time, for example. A multitude of applications of the sum-of-ratios problem can be envisioned in this way. Included is the case where some of the ratios are not proper quotients . This describes situations where a compromise is sought between absolute and relative terms like profit and return on investment (profit/capital) or return and return/risk, for example [49].

Almogy and Levin [1] analyze a multistage stochastic shipping problem. A deterministic equivalent of this stochastic problem is formulated which turns out to be a sum-of-ratios problem. For another presentation of this application see [23].

Rao [44] discusses various models in cluster analysis. The problem of optimal partitioning of a given set of entities into a number of mutually exclusive and exhaustive groups (clusters) gives rise to various mathematical programming problems depending on which optimality criterion is used. If the objective is to minimize the sum of the average squared distances within groups, then a minimum of a sum of ratios is to be determined.

The minimization of the mean response time in queueing location problems gives rise to (1.2) as well, as shown by Drezner, Schaible and Simchi-Levi [21]; see also [61].

Furthermore we mention an inventory model analyzed in [52] which is designed to determine simultaneously optimal lot sizes and an optimal storage allocation in a warehouse [27]. The total cost to be minimized is the sum of fixed cost per unit, storage cost per unit and material handling cost per unit.

In [34] Konno and Inori formulate a bond portfolio optimization problem as a sum-of-ratios problem.

More recently other applications of the sum-of-ratios problem have been identified. Mathis and Mathis [43] formulate a hospital fee optimization problem in this way. The model is used by hospital administrators in the State of Texas to decide on relative increases of charges for different medical procedures in various departments.

According to [14] a number of geometric optimization problems give rise to the sum-of-ratios problem. These often occur in layered manufacturing [41, 42, 54], for instance in material layout and cloth manufacturing [3]. For various examples we refer to the survey by Chen et al. [14] and the references therein. Quite in contrast to other applications of the sum-of-ratios problem mentioned before, the number of variables is very small (one, two or three), but the number of ratios is large; often there are hundreds or even thousands of ratios involved.

3 Preliminary theoretical results

As we know from single-ratio fractional programming [50], the maximization of a ratio of a (nonnegative) concave and a (positive) convex function is of particular interest in applications. Such a function is semistrictly quasiconcave, and hence a local is a global maximum [4]. In the differentiable case the ratio is a pseudoconcave function. Thus a Karush-Kuhn-Tucker point is a global maximum. Furthermore a single-ratio concave-convex fractional program can be related to a concave maximization problem with help of the generalized Charnes-Cooper transformation of variables [45]. This in turn provides a basis for introducing a dual and deriving duality relations for such single-ratio problems [45].

Naturally analogous results hold for the minimization of a convex-concave ratio.

Following the analysis of single-ratio fractional programs in the 60's and 70's, the multi-ratio case in the form of the max-min fractional program became the focus in the 80's. Here the smallest of a finite number of ratios is to be maximized. If in a max-min fractional program all ratios are concave-convex, then most of the properties of single-ratio problems mentioned above still hold in the multi-ratio case [50]. The only difference is that duality cannot be introduced through a transformation of variables. But various other approaches have been proposed leading to a rich duality theory for concave-convex max-min fractional programs;

see e.g., [15, 30].

Hence properties of concave-convex single-ratio fractional programs essentially extend to concave-convex max-min fractional programs. Such positive experience with the max-min case initially led to expectations that also sum-of-ratios problems might inherit the properties of single-ratio problems. However studies in the 90's showed that this is not so at all.

Unfortunately the sum of concave-convex ratios is generally not quasiconcave, not even the sum of a linear ratio and a linear function [46]. As a result a local is generally not a global maximum, even in the simple case mentioned before [46]. Bykadorov [8, 9, 10] has studied generalized concavity properties of sums of linear ratios and, more generally, of sums of ratios of polynomials.

Until now only a few properties are known for the concave-convex sum-of-ratios problem (1.1). Craven [17] shows that a maximum of the objective function is attained on the boundary of the convex feasible region. In case of two linear ratios an optimum is often (but not always) located in a vertex or on an edge of a polyhedral convex feasible region. For some additional properties we refer to the following section on algorithms; see also [12, 37, 32].

Recently Scott and Jefferson [55] proposed a first duality concept for (1.1) in case of linear ratios employing geometric programming duality. Freund and Jarre [24] showed that the sum-of-ratios problem is *essentially* $\mathcal{NP}$-complete, even in case of a sum of a concave-convex ratio and a concave function.

Although our understanding of the sum-of-ratios problem until now is quite limited, it is clear that in spite of the special structure of the objective function it is a much more challenging problem than the corresponding max-min fractional program. Given the small theoretical basis, it is not surprising that algorithmic advances have been rather limited too. However in recent years some progress has been made.

4 Algorithms

In this section we are outlining some algorithms which are of particular interest in solving the sum-of-ratios problem.

In [2] Almogy and Levin try to extend Dinkelbach's method [20] to the sum-of-ratios problem. The algorithm is based on decoupling numerators and denominators. Then the problem turns into the following:

$$H(q) = \max \left\{ \sum_{i=1}^{p} \Big(f_i(x) - q_i g_i(x) \Big) \;\middle|\; x \in S \right\} \tag{4.1}$$

where $q = (q_1, \cdots, q_p)$ is a vector of p parameters. However, Falk and Palocsay [23] give a numerical example showing that the Almogy-Levin algorithm does not work in general.

Cambini, Martein and Schaible [12] show that in a linear sum-of-ratios problem one ratio can be reduced to a linear function by applying the Charnes-Cooper transformation [13] used in single-ratio problems. Then Cambini, Martein and

Schaible [12] propose an algorithm for maximizing the sum of a linear function and a linear ratio:

$$\max\left\{h^\top x + (c^\top x+\alpha)/(d^\top x+\beta)\,\middle|\, x\in S\right\} \tag{4.2}$$

where c, d, h are vectors in R^n, $\alpha \in R, \beta \in R$ and $S = \{x \in R^n | Ax \le b, x \ge 0\}$ for some $(m \times n)$-matrix A and vector b of R^m. If the value of the denominator is fixed, then problem (4.2) turns into a linear program. Fixing the denominator to ξ, the authors solve the following problem [12]:

$$P(\xi) = \frac{1}{\xi}\max\left\{\xi h^\top x + c^\top x+\alpha \,\middle|\, x\in S, \xi = d^\top x+\beta\right\}. \tag{4.3}$$

Since S is a convex polyhedron, the parametric problem $P(\xi)$ attains its optimum on finitely many bases. As a result the algorithm solves problem (4.2) in finitely many iterations, even in case of an unbounded feasible region S. It either finds a global optimum or shows that the objective function is unbounded.

Falk and Palocsay [23] cast problem (1.1) into a linear maximization problem $\max\sum_{i=1}^p \varsigma_i$ over the image of the feasible region

$$\left\{(\varsigma_1,\cdots,\varsigma_p)\in R^p \,\middle|\, \varsigma_i(x) = \frac{f_i(x)}{g_i(x)},\ i=1,\cdots,p, x\in S\right\}. \tag{4.4}$$

Though some limited computational experience is available, a rigorous convergence proof of the proposed algorithm in [23] seems to be still lacking.

In [36] Konno, Kuno and Yajima use a parametric idea for solving the multiplicative program

$$\min\left\{m(x) = h(x) + \sum_{i=1}^p f_i(x)g_i(x) \,\middle|\, x\in S\right\} \tag{4.5}$$

where $f_i(x) > 0$, $g_i(x) > 0$ and $h(x)$ are convex functions on the compact convex set S. Consider

$$M(x,\xi,\eta) = h(x) + \sum_{i=1}^p\left(\frac{\xi_i(f_i(x))^2}{2} + \frac{\eta_i(g_i(x))^2}{2}\right)$$

where $\xi = (\xi_1,\cdots,\xi_p)$ and $\eta = (\eta_1,\cdots,\eta_p)$. Then the authors show that problem (4.5) is equivalent to the following so-called master program

$$\min\{M(x,\xi,\eta)\,|\,x\in S, \xi_i>0, \eta_i>0, \xi_i\eta_i\ge 1, i=1,\cdots,p\}. \tag{4.6}$$

Therefore an optimal solution (x^*,ξ^*,η^*) of (4.6) provides an optimal solution x^* for the original problem (4.5). To solve the program (4.6), the authors suggest to fix ξ and η in (4.6) and to solve the subproblem

$$\pi(\xi,\eta) = \min\{M(x,\xi,\eta)\,|\,x\in S\} \tag{4.7}$$

which is a convex minimization problem. An optimal solution (x^*, ξ^*, η^*) of (4.6) is yielded by solving

$$\min\{\pi(\xi,\eta) \,|\, \xi_i > 0, \eta_i > 0, \xi_i\eta_i \geq 1, i = 1, \cdots, p\}. \tag{4.8}$$

This problem is solved by an outer approximation method in [36].

Problem (4.5) is a generalized form of the sum-of-ratios problem. In fact, if we set

$$f_i(x) = c_i^\top x + \alpha_i, g_i(x) = \frac{1}{d_i^\top x + \beta_i},$$

then (4.5) becomes a linear sum-of-ratios problem. The assumptions for (4.5) hold since $g_i(x)$ is still convex. The linearity of $f_i(x)$ and $\frac{1}{g_i(x)}$ is not essential here. The linear ratios can be replaced by a convex over a concave function, and the method in [36] is still valid. Such a parametric approach for solving a multiplicative program reduces to the one in [35] where $p = 1$ and both $f_i(x)$ and $g_i(x)$ are linear in (4.5).

Konno and Fukaishi in [33] give a convex relaxation for problem (1.2). The relaxation is in the form of a linear programming problem with quadratic constraints $\sum_j \sum_k q_{jk}x_jx_k$. The authors consider an underestimating function of each nonconvex term $q_{jk}x_jx_k$. A bisection method is suggested for dividing a hyper-rectangle which contains the feasible region. The numerical experiments show that the proposed branch-and-bound algorithm is much faster than the algorithms proposed by Konno and Yamashita [38] and Falk and Palocsay [23].

The branch-and-bound approach for solving problem (1.1) via (4.4) is simple and powerful. A basic idea of such a branch-and-bound method is to create a p-dimensional box $[l, u]$ which contains an optimal solution ς^* of $\max_{i=1}^p \varsigma_i$ over (4.4). Then the box is divided into several smaller ones by a certain rule. After establishing a lower and upper bound on each smaller box, one can choose all the boxes with a potential of containing an optimal solution and discard the others which do not contain an optimal solution.

Dür, Horst and Thoai propose such a branch-and-bound algorithm in [22] for (1.1). Let Y^0 be a rectangle which contains the image set in (4.4). Then problem (1.1) is equivalent to the following problem

$$\max\left\{\sum_{i=1}^p \varsigma_i \,\middle|\, f_i(x) - \varsigma_i g_i(x) \geq 0, i = 1, \cdots, p, x \in S, \varsigma \in Y^0\right\}. \tag{4.9}$$

The authors' algorithm in [22] is based on systematically subdividing the rectangle Y^0. It is also discussed how one can obtain improved lower and upper bounds in case of affine ratios.

In [24] Freund and Jarre study problem (4.5) where $g_i(x)$ is replaced by $\frac{1}{g_i(x)}$. Here $g_i(x)$ is a concave function. For the case of $p = 1$ the authors introduce function $x(r)$ and $q(r)$ as follows

$$x(r) = \arg\min\left\{h(x) + \frac{f(x)}{r} \,\middle|\, g(x) \geq r \text{ and } x \in S\right\} \tag{4.10}$$

and

$$q(r) = h(x(r)) + \frac{f(x(r))}{r}. \tag{4.11}$$

It is obvious that $x(r^*)$ minimizes the original problem if and only if r^* minimizes $q(r)$. The authors suggest searching for an optimal solution by a branch-and-bound method for r in the interval

$$[0, \max\{g(x) \mid x \in S\}].$$

In a fixed interval, they use an interior-point method to obtain a lower bound. After the case of $p = 1$, they also extend their method to the general case of $p > 2$. Another main contribution in [24] is a proof that problem (4.5) is *essentially* $\mathcal{NP}$-complete.

Benson's algorithms in [5, 6, 7] are also based on branch-and-bound approaches. In [5], the following function is employed to solve (1.2)

$$\psi(y,z) = \frac{\left[\sum_{i=1}^{p} y_i \left(\prod_{j=1, j\neq i}^{p} z_i\right)\right]^{1/p}}{\left(\prod_{i=1}^{p} z_i\right)^{1/p}} \tag{4.12}$$

where $y = (y_1, \cdots, y_p)$ and $z = (z_1, \cdots, z_p)$. The numerator and denominator of ψ are artificially inflated and designed for computing lower and upper bounds hereafter. Indeed $\psi(y,z) = (\sum_{i=1}^{p} y_i/z_i)^{1/p}$. Let W and H be two subsets of R^{2p} given by

$$W = \left\{(y,z) \in R^{2p} \mid y_i = f_i(x), z_i = g_i(x) \text{ for some } x \in S\right\} \tag{4.13}$$

and

$$H = \left\{(y,z) \in R^{2p} \mid 0 \le y_i \le \max_{x\in S} f_i(x), 0 \le z_i \le \max_{x\in S} g_i(x)\right\}.$$

Then (1.2) is equivalent to the following problem:

$$\min\{\psi(y,z) \mid (y,z) \in W\}. \tag{4.14}$$

In [5] a branching process operates on H while a bounding process on $H \cap W$ works by virtue of the properties of the numerator and denominator of $\psi(\cdot,\cdot)$. In fact, $[\prod_{i=1}^{p} z_i]^{1/p}$ is a concave function on the interior set of R^{2p}. More precisely, the author estimates a lower bound $(N_H)^{1/p}$ for the numerator and an upper bound D_H for the denominator on $H \cap W$, respectively. Then he obtains a lower bound $\frac{(N_H)^{1/p}}{D_H}$ of $\psi(\cdot,\cdot)$ on $H \cap W$. In that sense both the branching and bounding are carried out in both axes y and z, respectively.

In [39] Kuno proposes a branch-and-bound method for problem (1.1). Though $S = \{Ax = b, x \ge 0\}$ for an $m \times n$ matrix A, a vector $b \in R^m$ and both $f_i(x)$

and $g_i(x)$ are linear, the algorithm works well for the nonlinear case too. For that reason we speak of Kuno's method within the framework of the nonlinear case as in problem (1.1). We use the following additional notation

$$\begin{array}{l} 0 < s_i^1 \leq \min\{f_i(x)/g_i(x) \,|\, x \in S\}, \\ \infty > t_i^1 \geq \max\{f_i(x)/g_i(x) \,|\, x \in S\}, \\ 0 < u_i \leq \min\{f_i(x) + g_i(x) \,|\, x \in S\}, \\ \infty > v_i \geq \max\{f_i(x) + g_i(x) \,|\, x \in S\}; \\ \Gamma_i = \{(y_i, z_i) \in R_+^2 \,|\, u_i \leq y_i + z_i \leq v_i\}, \\ \Delta_i^1 = \{(y_i, z_i) \in R_+^2 \,|\, s_i^1 z_i \leq y_i \leq t_i^1 z_i\}, \end{array} \tag{4.15}$$

where y and z are the same as in (4.13) and R_+ is the nonnegative orthant of R. Let

$$\Gamma = \prod_i^p \Gamma_i, \quad \Delta^1 = \prod_i^p \Delta_i^1.$$

It is easy to see that problem (1.1) reduces to

$$\max \left\{ z = \sum_{i=1}^p \frac{y_i}{z_i} \;\middle|\; (y, z) \in W \cap \Gamma \cap \Delta^1 \right\}. \tag{4.16}$$

Notice that Δ_i^1 is a cone and $\Gamma_i \cap \Delta_i^1$ is a trapezoid. Therefore $\Gamma_i \cap \Delta_i^1$ can be divided into two smaller trapezoids by a ray from the origin. In case of bisection, Δ_i^1 is partitioned into

$$\Delta_i^{1L} = \left\{ (y_i, z_i) \in R_+^2 \;\middle|\; s_i^1 z_i \leq y_i \leq \frac{s_i^1 + t_i^1}{2} z_i \right\}$$

and

$$\Delta_i^{1R} = \left\{ (y_i, z_i) \in R_+^2 \;\middle|\; \frac{s_i^1 + t_i^1}{2} z_i \leq y_i \leq t_i^1 z_i \right\}.$$

Such a division is very easy to implement just like the interval $[s_i^1, t_i^1]$ is divided into $\left[s_i^1, \frac{s_i^1+t_i^1}{2}\right]$ and $\left[\frac{s_i^1+t_i^1}{2}, t_i^1\right]$. We want to emphasize that such partitioning is carried out by changing the coefficient of z_i in Δ_i^1. Since it is a 1-dimensional operator, the division is cheap. Indeed, the algorithm in [39] may partition the cone Δ^1 into $\Delta^j = \prod_{i=1}^p \Delta_i^j$ by bisection as above. Notice that the four vertices of $\Gamma_i \cap \Delta_i$ can be calculated easily. The author uses the following two affine functions as overestimators of $\dfrac{y_i}{z_i}$. We see that the following two affine functions have the same values of $\frac{y_i}{z_i}$ of (4.16) at three of the four vertices, respectively

$$(t_i + 1)\frac{y_i - s_i z_i}{u_i} + s_i, \quad (s_i + 1)\frac{y_i - t_i z_i}{v_i} + t_i$$

where u_i, v_i define the vertices (see Fig.1 in [39]). Therefore

$$\phi_i(y, z) = \min \left\{ (t_i + 1)\frac{y_i - s_i z_i}{u_i} + s_i, (s_i + 1)\frac{y_i - t_i z_i}{v_i} + t_i \right\} \geq \frac{y_i}{z_i}$$

on $\Gamma_i \cap \Delta_i$ and can serve as the best concave overestimator of $\frac{y_i}{z_i}$. For a certain $\Gamma \cap \Delta$ the following concave minimization provides an upper bound on $\Gamma \cap \Delta$

$$\max \left\{ \sum_{i=1}^{p} \phi_i(y,z) \,\middle|\, (y,z) \in W \cap \Gamma \cap \Delta \right\}. \tag{4.17}$$

In [39] Kuno also reports some numerical results on the proposed algorithm SUMRATIO with varying $p = 1, \cdots, 6$. The algorithm SUMRATIO can be improved by a tighter bounding approach which only uses one well-selected affine function $\phi_i(y,z)$ [40]. The author reports that the improved algorithm is quite superior to the earlier one in [39]. CPU running times are reduced to almost one half for solving test problems.

Benson proposes two approaches in [6, 7] which were motivated by Kuno [39]. In [6], instead of solving the original problem (1.1), the author suggests to consider the following programming problem

$$\begin{aligned} v_H = \max \quad & \sum_{i=1}^{p} \frac{y_i}{z_i} \\ s.t. \quad & f_i(x) - y_i \geq 0, i = 1, 2, \cdots, p, \\ & -g_i(x) + z_i \geq 0, i = 1, 2, \cdots, p, \\ & x \in S, (y,z) \in H \end{aligned} \tag{4.18}$$

where H is a $2p$-dimensional box. The author points out that on a rectangle

$$\{(y_i, z_i) \in R^2 \,|\, \underline{y_i} \leq y_i \leq \overline{y_i}, \underline{z_i} \leq z_i \leq \overline{z_i}\}$$

the following minimization of affine functions is the concave envelope of $\frac{y_i}{z_i}$:

$$\min \left\{ \frac{1}{\underline{z_i}} y_i - \left(\frac{\underline{y_i}}{\underline{z_i}\overline{z_i}} \right) z_i + \frac{\underline{y_i}}{\overline{z_i}},\ \frac{1}{\overline{z_i}} y_i - \left(\frac{\overline{y_i}}{\underline{z_i}\overline{z_i}} \right) z_i + \frac{\overline{y_i}}{\underline{z_i}} \right\}. \tag{4.19}$$

With this concave envelope, the author proposes an algorithm in [6] in which upper bounds on a subregion are obtained by maximizing (4.19) over the subregion and the branching process takes place only in the p-dimensional space $(z_1, \cdots, z_p)$.

In [7], the author suggests solving

$$\begin{aligned} \max \quad & \sum_{i=1}^{p} y_i f_i(x) \\ s.t. \quad & y_i g_i(x) - 1 \leq 0, i = 1, 2, \cdots, p, \\ & x \in S, y \in \left\{ y \in R^p \,\middle|\, \frac{1}{\max_{x \in S} g_i(x)} \leq y_i \leq \frac{1}{\min_{x \in S} g_i(x)} \right\}. \end{aligned} \tag{4.20}$$

which is equivalent to the original problem (1.1). Obviously, on a p-dimensional box $\{y \in R^p \,|\, \underline{y} \leq y \leq \overline{y}\}$ an optimum of the following programming problem provides a local upper bound of (4.20) on the box

$$\begin{aligned} \max \quad & \sum_{i=1}^{p} \overline{y}_i f_i(x) \\ s.t. \quad & \underline{y}_i g_i(x) - 1 \leq 0, \ i = 1, 2, \cdots, p, \\ & x \in S. \end{aligned} \tag{4.21}$$

Note that problem (4.21) is a maximization of a concave function over a convex set. Therefore algorithms based on dividing the p-dimensional box

$$\left[\frac{1}{\max_{x\in S} g_i(x)},\ \frac{1}{\min_{x\in S} g_i(x)}\right]$$

and solving (4.21) can be designed for the original problem (1.1). Though there are some numerical examples illustrating the proposed algorithms in both [6] and [7], no computational results are reported in the two papers.

In [18] Dai and Shi consider a more general case of problem (1.1) where both numerators and denominators are d.c. functions [26, 29]. The authors transfer problem (1.1) to a linear programming problem over a union of reverse convex sets and solve the problem by using an algorithm in [19] which is designed for solving a multiple reverse convex problem. The approach can be considered as an extension of the method in [57].

5 Conclusion

The sum-of-ratios fractional program has important applications in several areas such as production, transportation, finance, engineering, statistics, for example. Unfortunately the current understanding of the structural properties of this challenging problem is quite limited. On the other hand, in recent years a number of interesting solution methods have been proposed. Some of these have been computationally tested. Typically execution times grow very rapidly in the number of ratios. At this time problems with up to about ten ratios can be handled. We refer here to the algorithms by Konno-Fukaishi [33] (see also [32]) and by Kuno [39, 40]. The former is superior to several earlier methods (see [33]) while the latter is seemingly faster than the former.

Some of the other proposed methods have been tested for a smaller number of ratios (e.g. [22, 24]). It seems that a more thorough testing of various algorithms is needed before further conclusions can be drawn. Also some of the applications call for methods which can handle a larger number of ratios, e.g., fifty [1]. Currently such methods are not available.

For a special class of sum-of-ratios problems with up to about one thousand ratios, but only very few variables an algorithm is given in [14]. This method by Chen et al. is superior to the other algorithms on the particular class of problems in manufacturing. These are geometric optimization problems arising in layered manufacturing. The linear case is considered in [14]. Related layered manufacturing problems with nonlinear ratios are found in [41, 54], again involving only a few variables, but a large number of ratios. In contrast to the general-purpose algorithms reviewed in this survey, the method in [14] is rather robust with regard to the number of ratios.

References

[1] Y. Almogy and O. Levin, Parametric analysis of a multi-stage stochastic shipping problem, in J. Lawrence (ed.) *Operational Research'69*, 359-370, Tavistock Publications, London, 1970.

[2] Y. Almogy and O. Levin, A class of fractional programming problems, *Operations Research* **19** (1971) 57-67.

[3] E.M. Arkin, Y.-J. Chiang, M. Held, J.S.B. Mitchel, S. Sacristan, S.S. Skiena and T.-C. Yang, On minimum area hulls, *Algorithmica* **21** (1998) 119-136.

[4] M. Avriel, W.E. Diewert, S. Schaible and I. Zang, *Generalized concavity*, Mathematical Concepts and Methods in Science and Engineering **36**, Plenum Press, New York, 1988.

[5] H.P. Benson, Global optimization of nonlinear sums of ratios, *Journal of Mathematical Analysis and Applications* **263** (2001) 301-315.

[6] H.P. Benson, Using concave envelopes to globally solve the nonlinear sum of ratios problem, *Journal of Global Optimization* **22** (2002) 343-364.

[7] H.P. Benson, Global optimization algorithm for the nonlinear sum of ratios problem, *Journal of Optimization Theory and Applications* **112** (2002) 1-29.

[8] I.A. Bykadorov, Conditions for the quasiconvexity of sums of linear-fractional functions, *Optimizatsiya* **39** (1986) 25-41, 158. (Russian)

[9] I.A. Bykadorov, Quasiconvexity criteria for objective functions in fractional programming problems, *Doctoral Thesis in Philosophy (Mathematics)* , Institut Matematiki, Novosibirsk, 1990. (Russian)

[10] I.A. Bykadorov, On quasiconvexity in fractional programming, in S. Komlosi, T. Rapcsak and S. Schaible (eds.), *Generalized convexity*, Lecture Notes in Economics and Mathematical Systems **405**, 281-293, Springer, Berlin, 1994.

[11] A. Cambini, E. Castagnoli, L. Martein, P. Mazzoleni and S. Schaible (eds.), *Generalized convexity and fractional programming with economic applications.* Lecture Notes in Economics and Mathematical Systems **345**, Springer, Berlin, 1990.

[12] A. Cambini, L. Martein and S. Schaible, On maximizing a sum of ratios, *Journal of Information and Optimization Sciences* **10** (1989) 65-79.

[13] A. Charnes and W.W. Cooper, Programming with linear fractional functionals, *Naval Research Logistics Quarterly* **9** (1962) 181-186.

[14] D.Z. Chen, O. Daescu, Y. Dai, N. Katoh, X. Wu and J. Xu, Optimizing the sum of linear fractional functions and applications. *Proceedings of the Eleventh Annual ACM-SIAM Symposium on Discrete Algorithms*, 707-716, ACM, New York, 2000.

[15] J.P. Crouzeix, J.A. Ferland and S. Schaible, Duality in generalized linear fractional programming, *Mathematical Programming* **27** (1983) 342–354.

[16] J.P. Crouzeix, J.E. Martinez-Legaz and M. Volle (eds.), *Generalized convexity, generalized monotonicity: recent results*, Kluwer Academic Publishers, Dordrecht, 1998.

[17] B.D. Craven, *Fractional Programming*, Sigma Series in Applied Mathematics **4**, Heldermann Verlag, 1988.

[18] Y. Dai and J. Shi, A conical partition algorithm for maximizing the sum of several dc ratios, *Discussion Paper* **MS-01-06** , School of Management, Tokyo University of Science, 2001.

[19] Y. Dai, J. Shi and Y. Yamamoto, Global optimization problem with several reverse convex constraints and its application to out-of-roundness problem, *Journal of the Operations Research Society of Japan* **39** (1996) 356-371.

[20] W. Dinkelbach, On nonlinear fractional programming, *Management Science* **13** (1967) 492-498.

[21] Z. Drezner, S. Schaible and D. Simchi-Levi, Queueing-location problems on the plane, *Naval Research Logistics* **37** (1990) 929-935.

[22] M. Dür, R. Horst and N.V. Thoai, Solving sum-of-ratios fractional programs using efficient points, *Optimization* **49** (2001) 447–466.

[23] J.E. Falk and S.W. Palocsay, Optimizing the sum of linear fractional functions, in C.A. Floudas and P.M. Pardalos (eds.), *Recent Advances in Global Optimization*, 221-258, Kluwer Academic Publishers, Dordrecht, 1992.

[24] R.W. Freund and F. Jarre, Solving the sum-of-ratios problem by an interior-point method, *Journal of Global Optimization* **19** (2001) 83-102.

[25] N. Hadjisavvas, J.E. Martinez-Legaz and J.P. Penot (eds.), *Generalized convexity and generalized monotonicity*, Lecture Notes in Economics and Mathematical Systems **502**, Springer, Berlin, 2001.

[26] P. Hartman, On functions representable as a difference of convex functions, *Pacific Journal of Mathematics* **9** (1959) 707-713.

[27] T.J. Hodgson and T.J. Lowe, Production lot sizing with material handing cost considerations, *IIE Transactions* **14** (1982) 44-51.

[28] R. Horst and P.M. Pardalos (eds.), *Handbook of Global Optimization*, Kluwer Academic Publishers, Dordrecht, 1995.

[29] R. Horst and H. Tuy, *Global Optimization: Deterministic Approaches*, Springer, Berlin 1993.

[30] R. Jagannathan and S. Schaible, Duality in generalized fractional programming via Farkas' lemma, *Journal of Optimization Theory and Applications* **41** (1983) 417–424.

[31] S. Komlosi, T. Rapcsak and S. Schaible (eds.), *Generalized convexity*, Lecture Notes in Economics and Mathematical Systems **405**, Springer, Berlin, 1994.

[32] H. Konno, Minimization of the sum of several linear fractional functions, in N. Hadjisavvas, J.E. Martinez-Legaz and J.P. Penot (eds.), *Generalized convexity and generalized monotonicity*, Lecture Notes in Economics and Mathematical Systems **502**, 3-20, Springer, Berlin, 2001.

[33] H. Konno, and K. Fukaishi, A branch and bound algorithm for solving low rank linear multiplicative and fractional programming problems, *Journal of Global Optimization* **18** (2000) 283-299.

[34] H. Konno, and M. Inori, Bond portfolio optimization by bilinear fractional programming, *Journal of the Operations Research Society of Japan* **32** (1989) 143-158.

[35] H. Konno and T. Kuno, Generalized linear multiplicative and fractional programming, *Annals of Operations Research* **25** (1990) 147-161.

[36] H. Konno and T. Kuno and Y. Yajima, Global minimization of a generalized convex multiplicative function, *Journal of Global Optimization* **4** (1994) 47-62.

[37] H. Konno, P.T. Thach and H. Tuy, *Optimization on Low Rank Nonconvex Structures*, Kluwer Academic Publishers, Dordrecht, 1997.

[38] H. Konno and H. Yamashita, Minimization of the sum and the product of several linear fractional functions, *Naval Research Logistics* **46** (1999) 583-596.

[39] T. Kuno, A branch-and-bound algorithm for maximizing the sum of several linear ratios, *Journal of Global Optimization* **22** (2002) 155-174.

[40] T. Kuno, On deterministic algorithms for the sum-of-ratios problem, *Private communication*, November, 2002.

[41] J. Majhi, R. Janardan, J. Schwerdt, M. Smid and P. Gupta, Minimizing support structures and trapped areas in two-dimensional layered manufacturing, *Computational Geometry* **12** (1999) 241-267.

[42] J. Majhi, R. Janardan, M. Smid and P. Gupta, On some geometric optimization problems in layered manufacturing, *Computational Geometry* **12** (1999) 219-239.

[43] F.H. Mathis and L.J. Mathis, A nonlinear programming algorithm for hospital management, *SIAM Review* **37** (1995) 230-234.

[44] M.R. Rao, Cluster analysis and mathematical programming, *Journal of the American Statistical Association* **66** (1971) 622-626.

[45] S. Schaible, Fractional programming I: duality, *Management Science* **22** (1976) 858-867.

[46] S. Schaible, A note on the sum of a linear and linear-fractional function, *Naval Research Logistics Quarterly* **24** (1977) 691-693.

[47] S. Schaible, *Analyse und Anwendungen von Quotientenprogrammen*, *Ein Beitrag zur Planung mit Hilfe der nichtlinearen Programmierung*, Mathematical Systems in Economics **42**, Hain-Verlag, Meisenheim, 1978.

[48] S. Schaible, Fractional programming: applications and algorithms, *European Journal of Operational Research* **7** (1981) 111-120.

[49] S. Schaible, Simultaneous optimization of absolute and relative terms, *Zeitschrift für Angewandte Mathematik und Mechanik* **64** (1984) 363-364.

[50] S. Schaible, Fractional programming, in R. Horst and P.M. Pardalos (eds.), *Handbook of Global Optimization*, Kluwer Academic Publishers, Dordrecht, 1995, 495-608.

[51] S. Schaible and T. Ibaraki, Fractional programming, invited review, *European Journal of Operational Research* **12** (1983) 325-338.

[52] S. Schaible and T. Lowe, A note on a material control problem, *IIE Transactions* **15** (1983) 177-179.

[53] S. Schaible and W.T. Ziemba (eds.), *Generalized concavity in optimization and economics*, Academic Press, New York, 1981.

[54] J. Schwerdt, M. Smid, R. Janardan, E. Johnson and J. Majhi, Protecting critical facets in layered manufacturing, *Computational Geometry* **16** (2000) 187-210.

[55] C. H. Scott and T. R. Jefferson, Duality of a nonconvex sum of ratios, *Journal of Optimization Theory and Applications* **98** (1998) 151–159.

[56] H. Sherali and H. Wang, Global optimization of nonconvex factorable programming problems, *Mathematical Programming* **89** (2001) 459-478.

[57] J. Shi, A combined algorithm for fractional programming, *Annals of Operations Research* **103** (2001) 135-147.

[58] C. Singh and B.K. Dass (eds.), Continuous-time, fractional and multiobjective programming, Proceedings of a Conference at St. Lawrence University, Canton, NY, 1986, *Journal of Information and Optimization Sciences* **10** (1989) No.1 (January).

[59] M. Sniedovich, A new look at fractional programming, *Journal of Optimization Theory and Applications* **54** (1987) 113-120.

[60] I.M. Stancu-Minasian, *Fractional Programming*, Kluwer Academic Publishers, Dordrecht, 1997.

[61] S. Zhang, Stochastic Queue Location Problems, *Doctoral Dissertation*, Econometric Institute, Erasmus University, Rotterdam, 1991.